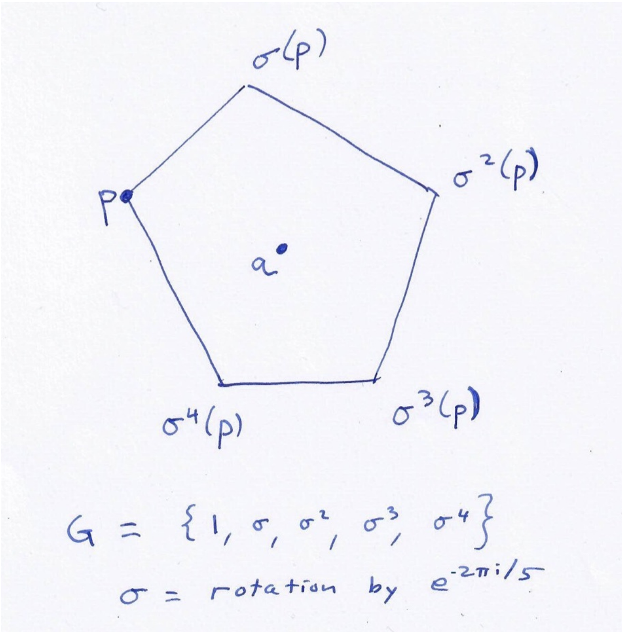
A finite group of isometries of $\mathbb{R}^n$ fixes a point

Theorem. *Let G be a finite group of isometries of $\mathbb{R}^n$. Then there exists $a \in \mathbb{R}^n$ such that $\phi(a) = a$ for all $\phi \in G$.*

The idea of the proof is this. Let p be any point in $\mathbb{R}^n$, and let

$$G \cdot p := \{\phi(p) \mid \phi \in G\}$$

be the orbit of p under the action of G. Define the point a to be the *center of gravity* (the average) of the points in $G \cdot p$, as illustrated in the figure for the group C_5 acting on $\mathbb{R}^2$.



We claim: Every element of G fixes a.

To prove this, we will establish a general principle: An isometry ψ carries the center of gravity of a figure to the center of gravity of its image.

We need the fact that isometries are affine (linear plus a constant).

Hilfsatz. (Proof elsewhere) *Every isometry ψ of $\mathbb{R}^n$ has the form*

$$\psi(x) = Ax + b,$$

where A is an orthogonal matrix and $b \in \mathbb{R}^n$.

Now let $x_1, \dots, x_m$ be any finite point set. Define

$$\mathrm{CG}(x_1, \dots, x_m) := \frac{x_1 + \cdots + x_m}{m}.$$

Compute

$$\begin{aligned}
\psi(\mathrm{CG}(x_1, \dots, x_m)) &= \psi\left(\frac{x_1 + \cdots + x_m}{m}\right) \\
&= A\left(\frac{x_1 + \cdots + x_m}{m}\right) + b \\
&= \frac{(Ax_1 + b) + \cdots + (Ax_m + b)}{m} \\
&= \frac{\psi(x_1) + \cdots + \psi(x_m)}{m} \\
&= \mathrm{CG}(\psi(x_1), \dots, \psi(x_m)).
\end{aligned}$$

This establishes the geometric invariance of the center of gravity.

Proof of Theorem. Let p be any point in $\mathbb{R}^n$. Let a be the center of gravity of the orbit of p, namely

$$a := \frac{\sum_{\phi \in G} \phi(p)}{|G|}.$$

Let $\psi \in G$ be arbitrary. Compute

$$\begin{aligned}
\psi(a) &= \psi\left(\frac{\sum_{\phi \in G} \phi(p)}{|G|}\right) \\
&= \frac{\sum_{\phi \in G} \psi(\phi(p))}{|G|} \\
&= \frac{\sum_{\phi \in G} \phi(p)}{|G|} \\
&= a.
\end{aligned}$$

In the second line, we used the geometric invariance of the center of gravity. In the third line, we used the fact that

$$\psi \circ \phi, \qquad \phi \in G,$$

is just a reordering of

$$\phi, \qquad \phi \in G.$$

□