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Assignment 7

Groups, Subgroups, Group Homomorphism

1. Prove that the map $f : \mathbb{R} \longrightarrow \mathbb{C}^\times$, defined by $f(x) := e^{ix}$ is a group homomorphism. Find its kernel and its image.

2. Find the order of the following elements:

   (a) i, $e^{i\sqrt{3}\pi}$ and $e^{\frac{2\pi i}{17}}$ in the group $\mathbb{C}^\times$;

   (b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ in the group $\mathrm{GL}_2(\mathbb{C})$;

   (c) $1, 2$ and 3 in $\mathbb{F}_{17}^\times$.

3. Let p be a prime number. Show that the cardinality of $\mathrm{GL}_2(\mathbb{F}_p)$ is equal the number of ordered bases (e_1, e_2) of $\mathbb{F}_p^2$ as a $\mathbb{F}$-vector space, and that

$$\mathrm{Card}(\mathrm{GL}_2(\mathbb{F}_p)) = (p-1)^2 p(p+1).$$

4. Let $\mathcal{C}$ be a category.

   (a) For an object A of $\mathcal{C}$ let $\mathrm{Aut}_\mathcal{C}(A)$ be the set of isomorphisms from A to A, i.e.

$$\mathrm{Aut}_\mathcal{C}(A) = \{f \in \mathrm{Hom}_\mathcal{C}(A, A) : f \text{ is an isomorphism}\}.$$

   Let $f \circ g$ be the composition of morphisms $f, g : A \to A$ and let $\mathrm{id}_A \in \mathrm{Hom}_\mathcal{C}(A, A)$ be the identity homomorphism. Show that $(\mathrm{Aut}_\mathcal{C}(A), \circ, \mathrm{id}_A)$ is a group.

   Remark: For $\mathfrak{Set}$ the category of sets with homomorphisms being maps between sets, one has the object $A = \{1, 2, \ldots, n\}$, a finite set, and

$$\mathrm{Aut}_{\mathfrak{Set}}(A) = S_n$$

   is the symmetric group.

   (b) Let A, B isomorphic objects of $\mathcal{C}$. Show that the groups $\mathrm{Aut}_\mathcal{C}(A)$ and $\mathrm{Aut}_\mathcal{C}(B)$ are isomorphic.

5. Let $G = \mathrm{GL}_2(\mathbb{F}_2)$ and consider the set $X = (\mathbb{F}_2)^2 \smallsetminus \{(0,0)\}$. Define

$$H := \mathrm{Sym}(X) := \mathrm{Aut}_{\mathfrak{Set}}(X) = \{f : X \to X : f \text{ bijective}\}.$$

(a) Prove that

$$\begin{aligned} \varphi : G &\longrightarrow H \\ \alpha &\longmapsto (P \mapsto \alpha(P)) \end{aligned}$$

is a well-defined group homomorphism.

(b) Show that φ is an group isomorphism

(c) Deduce that $G \cong S_3$.

6. Let p be a prime number. Consider the set

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p) \right\} \subset \mathrm{GL}_2(\mathbb{F}_p).$$

(a) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$.

(b) Prove that the map

$$\begin{aligned} \varphi : G &\longrightarrow \mathbb{F}_p^\times \times \mathbb{F}_p^\times \\ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &\longmapsto (a, c) \end{aligned}$$

is a group homomorphism, where $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ is endowed with componentwise multiplication, and that $\ker(\varphi) \cong (\mathbb{F}_p, +)$.

7. Let $G = \mathrm{GL}_2(\mathbb{Q})$ and consider its elements $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Show that $A^4 = \mathrm{Id}_2 = B^6$, but that $(AB)^n \neq \mathrm{Id}_2$ for each $n \geqslant 1$.