Assignment 9

Group actions

1. Let G be a group. Consider the set of maps $C(G) = \{f : G \longrightarrow \mathbb{C}\}$.
   (a) Check that G acts on $C(G)$ via $(g \cdot f)(x) := f(xg)$ for $g, x \in G$ and $f \in C(G)$.
   (b) Is the action above faithful?

2. Let G be a group acting on a set T and $t_1, t_2 \in T$ be elements in the same G-orbit. Prove that the stabilizers of t_1 and t_2 in G are conjugate.

3. Let G be a group acting on a set T. For $H \subseteq G$, define the set of H-*invariants* as
$$T^H := \{x \in T \,:\, \forall h \in H, h \cdot x = x\}.$$
Prove: if $H \trianglelefteq G$, then the action of G on T induces an action of G/H on T^H.

4. Consider the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
   (a) Show that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$
   (b) Is the action faithful?
   (c) Show that the subgroup $H := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$ acts transitively on $\mathbb{H}$.
   (d) Compute the stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$.
   (e) Deduce that any $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as $g = hk$ for $h \in H$ and $k \in \mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$.
   (f) Compute and sketch the orbit of $i \in \mathbb{H}$ under the following subgroups:
$$H_1 := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\},\ H_2 := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\},\ H_3 := \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\}.$$

5. Let G be a finite group and $H \subset G$ a subgroup. Suppose that the index of H in G is equal to the smallest prime number dividing $|G|$. Prove: $H \triangleleft G$. [*Hint:* Define a suitable action $\rho : G \longrightarrow \mathrm{Sym}(G/H)$. Look at $\ker(\rho)$ and $\mathrm{Card}(\mathrm{Im}(\rho))$.]

6. Let G be a finite group and p a prime number. Let $\mathcal{T}_p$ be the set of all p-Sylow subgroups and fix $P \in \mathcal{T}_p$. Since conjugation preserves cardinality of subsets, G acts on $\mathcal{T}_p$ by

$$g \cdot H = gHg^{-1}.$$

   (a) Show that the induced action of P on $\mathcal{T}_p$ has a unique fixed point.

   (b) Deduce that $\mathrm{Card}(\mathcal{T}_p) \equiv 1 \pmod{p}$.

   (c) Prove that $\mathrm{Card}(\mathcal{T}_p) \mid m := [G : P]$. [*Hint:* Use the action of G by conjugation on the set of its subgroups]

   (d) Let $M \supset P$ be a subgroup of G containing $N_G(P)$. Prove that $N_G(M) = M$.

7. Let K be a field and D be the subgroup of $G := \mathrm{GL}_2(K)$ consisting of diagonal matrices. Determine $N_G(D)$ and $N_G(D)/D$.

8. Let S_n act on $\{1, \dots, n\}$. Define an action of S_n on $\{1, \dots, n\} \times \{1, \dots, n\}$ by $g \cdot (i, j) = (g(i), g(j))$. Show that this action has exactly two orbits and determine them.