Assignment 10

Group actions, the symmetric Group

1. Let p be a prime number and T the set of one-dimensional $\mathbb{F}_p$-subspaces in $(\mathbb{F}_p)^{n+1}$, i.e., of lines through the origin in $(\mathbb{F}_p)^{n+1}$.

   (a) Show that $\mathrm{GL}_{n+1}(\mathbb{F}_p)$ acts transitively on T by $g \cdot L = g(L)$.

   (b) Compute the stabilizer of the line $L_0 := \langle(1, 0, \dots, 0)\rangle \in T$.

   (c) Compute $\mathrm{Card}(T)$. [*Hint:* T has the same number of elements of the set of orbits of $\mathbb{F}_p^\times$ acting on $(\mathbb{F}_p)^{n+1} \smallsetminus \{0\}$]

2. Consider the standard action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$. Determine the orbits of $(1, 0)$ under each of the subgroups

$$H_1 := \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\},\ H_2 := \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\},\ H_3 := \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\},\ H_4 := \mathrm{SO}_2(\mathbb{R}).$$

3. Let G be a group acting on a set T. Fix $x_0 \in T$. Let $H \subset G$ be a subgroup and define X to be the H-orbit of x_0. Show that, for $g \in G$,

$$g \cdot X = \{g \cdot x : x \in X\}$$

   is the gHg^{-1}-orbit of $g \cdot x_0$.

4. Let $\sigma \in S_n$. Denote by $F(\sigma)$ the number of points fixed by σ. Prove that the following formulas hold:

$$\frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma) = 1$$
$$\frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma)^2 = 2$$

   [*Hint:* Notice that $F(\sigma) = \sum_{x:\sigma(x)=x} 1$. Invert the order of summation.]

5. For each conjugacy class S_6, write down a representative and the cardinality of the class.

6. Let $n \geqslant 3$. Prove that $[S_n, S_n] = A_n$. [Recall: for a group G, the commutator $[G, G]$ is defined as the subgroup of G generated by $\{aba^{-1}b^{-1} : a, b \in G\}$. See Assignment 8, Exercise 6]

7. (a) Prove that S_n is generated by $\{\sigma_i := (i \quad i+1), 1 \leqslant i \leqslant n-1\}$, and that those generators satisfy the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i-j| \geqslant 2$$
$$(\sigma_i \sigma_{i+1})^3 = \text{id, for } 1 \leqslant i \leqslant n-2.$$

   (b) Let $\tau := (1\,2\,\dots\,n)$. Show that S_n is generated by $\{\sigma_1, \tau\}$. [*Hint:* Express σ_i in terms of σ_1 and τ]

8. Let $n \geqslant 2$ be an integer and $k_i \in \mathbb{Z}_{\geqslant 0}$ for $i = 1, \dots, n$ be such that

$$k_1 \cdot 1 + k_2 \cdot 2 + \ldots + k_n \cdot n = n.$$

Let X be the conjugacy class of X determined by $(k_1, \dots, k_n)$. Tautologically, S_n acts on X by conjugation and the action is transitive.

   (a) Fix $\sigma_0 \in X$ and let $H = \mathrm{Stab}_{S_n}(\sigma_0)$. Prove that

$$\mathrm{Card}(H) = \prod_{i=1}^{n} i^{k_i} \cdot k_i!$$

   (b) Use the above to write down an expression for $\mathrm{Card}(X)$.

   (c) Show that $\mathrm{Card}(\{n - \text{cycles in } S_n\}) = (n-1)!$