D-MATH Algebra I HS18
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Solution 9

Group actions

1. Let G be a group. Consider the set of maps $C(G) = \{f : G \longrightarrow \mathbb{C}\}$.

   (a) Check that G acts on $C(G)$ via $(g \cdot f)(x) := f(xg)$ for $g, x \in G$ and $f \in C(G)$.

   (b) Is the action above faithful?

   Solution:

   (a) First, notice that $(g, f) \mapsto g \cdot f$ defines a map $G \times C(G) \longrightarrow C(G)$. This is compatible with the group structure since, for all $g, h \in G$ and $f \in C(G)$,

   $$\forall x \in G,\ ((gh) \cdot f)(x) = f(xgh) = (h \cdot f)(xg) = (g \cdot (h \cdot f))(x),$$

   implying that $(gh) \cdot f = g \cdot (h \cdot f)$. Moreover, $e_G \in G$ acts trivially on $C(G)$, since for all $f \in C(G)$

   $$\forall x \in X : (1_G \cdot f)(x) = f(x1_G) = f(x).$$

   so that $1_G \cdot f = f$. This proves that we have a group action

   (b) Consider the characteristic function $\chi : G \longrightarrow \mathbb{C}$ sending $1_G \mapsto 1_{\mathbb{C}}$ and $1_G \neq g \mapsto 0$. Then if $g \in \mathrm{Stab}_G(\chi)$, one can write $\chi(g) = g \cdot \chi(1_G) = \chi(1_G) = 1_{\mathbb{C}}$, which implies that if and only $g = 1_G$. This proves that $\mathrm{Stab}_G(\chi) = \{1_G\}$ and that the action is faithful.

2. Let G be a group acting on a set T and $t_1, t_2 \in T$ be elements in the same G-orbit. Prove that the stabilizers of t_1 and t_2 in G are conjugate.

   Solution: Since t_1, t_2 are in the same G-orbit, there exists $g \in G$ such that $t_2 = g_0 \cdot t_1$. Then

   $$\begin{aligned}\mathrm{Stab}_G(t_2) &= \{g \in G : g \cdot t_2 = t_2\} = \{g \in G : g \cdot (g_0 \cdot t_1) = g_0 \cdot t_1\} \\ &= \{g \in G : (g_0^{-1} g g_0) \cdot t_1 = t_1\} = \{g \in G : (g_0^{-1} g g_0) \in \mathrm{Stab}_G(t_1)\} \\ &= g_0 \mathrm{Stab}_G(t_1) g_0^{-1},\end{aligned}$$

   so that the stabilizers of t_1 and t_2 are conjugate.

3. Let G be a group acting on a set T. For $H \subseteq G$, define the set of H-*invariants* as

$$T^H := \{x \in T \, : \, \forall h \in H, h \cdot x = x\}.$$

Prove: if $H \trianglelefteq G$, then the action of G on T induces an action of G/H on T^H.

Solution: Consider the map

$$\begin{aligned} l : G/H \times T^H &\longrightarrow T^H \\ (gH, x) &\longmapsto g \cdot x. \end{aligned}$$

We first check that it is well-defined:

- For every $g \in G$, $x \in T^H$ and $h \in H$, we have that $h \cdot (g \cdot x) = g \cdot ((g^{-1}hg) \cdot x) = g \cdot x$ as $H \trianglelefteq G$. Hence $g \cdot x \in T^H$.
- If $gH = g'H$, then $g = g'h$ for $h \in H$. Then $g \cdot x = (g'h) \cdot x = g' \cdot (h \cdot x) = g' \cdot g$ for every $x \in T^H$, so that $\ell(gH)$ does not depend on the representative g.

The map l is an action of G/H on T^H, as the axioms of group actions are inherited from the given action of G on T.

4. Consider the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

   (a) Show that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

   (b) Is the action faithful?

   (c) Show that the subgroup $H := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$ acts transitively on $\mathbb{H}$.

   (d) Compute the stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$.

   (e) Deduce that any $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as $g = hk$ for $h \in H$ and $k \in \mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$.

   (f) Compute and sketch the orbit of $i \in \mathbb{H}$ under the following subgroups:

   $$H_1 := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \; H_2 := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}, \; H_3 := \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\}.$$

Solution:

(a) First, we check that for each $M \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$ we have $M \cdot z \in \mathbb{H}$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the denominator $cz + d$ has imaginary part cz, which is zero only if $c = 0$, in which case $d \neq 0$ because $M \in \mathrm{SL}_2(\mathbb{R})$. This means that $cz+d \neq 0$ so that $M \cdot z$ is a well-defined complex number. Writing $\alpha = \Re(z)$, $\beta = \Im(z)$, we see that

$$\begin{aligned} \Im(M \cdot z) &= \Im\left(\frac{a\alpha + ia\beta + b}{c\alpha + ic\beta + d}\right) = \Im\left(\frac{(a\alpha + ia\beta + b)(c\alpha - ic\beta + d)}{(c\alpha + d)^2 + c^2\beta^2}\right) \\ &= \frac{a\beta(c\alpha + d) - c\beta(a\alpha + b)}{(c\alpha + d)^2 + c^2\beta^2} = \frac{\det(A)}{(c\alpha + d)^2 + c^2\beta^2} = \frac{1}{(c\alpha + d)^2 + c^2\beta^2}. \end{aligned}$$

Hence $M \cdot x \in \mathbb{H}$. Hence we have indeed a map $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H}$. Now consider another $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then

$$M'M = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}, \ (M'M) \cdot z = \frac{(a'a + b'c)z + (a'b + b'd)}{(c'a + d'c)z + (c'b + d'd)},$$

whereas

$$M' \cdot (M \cdot z) = \frac{a'(M \cdot z) + b'}{c'(M \cdot z) + d'} = \frac{a'\frac{az+b}{cz+d} + b'}{c'\frac{az+b}{cz+d} + d'} = \frac{a'(az + b) + b'(cz + d)}{c'(az + b) + d'(cz + d)}$$

and we realise that $M' \cdot (M \cdot z) = (M'M) \cdot z$. Finally, the identity matrix maps $z \mapsto z$ by definition. This prove that we are indeed dealing with a groups action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.

(b) The equality $z = M \cdot z$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equivalent to

$$cz^2 + (d - a)z - b = 0.$$

We want to find for which M the above equality is satisfied for all $z \in \mathbb{Z}$. Substituting $z = i$ and comparing real and imaginary part, we obtain $-c-b = 0$ and $a = d$. Then, substituting $1 + i$, we obtain $2ic - b = 0$, which implies $b = c = 0$. Since $M \in \mathrm{SL}_2(\mathbb{R})$, we need that $a = d = \pm 1$. This shows that the only matrices fixing all elements of $\mathbb{H}$ are Id_2 and $-\mathrm{Id}_2$.

(c) In order to prove that the subgroup H acts transitively on $\mathbb{H}$, it is enough to show that the orbit of i contains all $w \in \mathbb{H}$. This amounts to show that for each $w = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$, there exists $a, b, d \in \mathbb{R}$, with $ad = 1$, such that

$$\alpha + i\beta = w = \frac{ai + b}{d}.$$

This can be attained (in a unique way) by taking $a = \sqrt{\beta}$, $d = \frac{1}{\sqrt{\beta}}$ and $b = \frac{\alpha}{\sqrt{\beta}}$.

(d) We compute the desidered stabilizer:

$$
\begin{aligned}
\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) &= \{M \in \mathrm{SL}_2(\mathbb{R}) : i = M \cdot i\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : di - c = ai + b \right\} = \left\{ \begin{pmatrix} a & c \\ -c & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\} \\
&= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R} \right\} = \mathrm{SO}_2(\mathbb{R}).
\end{aligned}
$$

(e) Let $g \in \mathrm{SL}_2(\mathbb{R})$. By part (c), there exists $h \in H$ such that $h \cdot i = g \cdot i$. Then $h^{-1}g \cdot i = i$, so that $k := h^{-1}g \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) = \mathrm{SO}_2(\mathbb{R})$. Then $g = hk$ is the desired composition.

(f) We compute the image of i under a general element of the different subgroups:

- Let $M = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in H_1$. Then $M \cdot i = \frac{ai}{a^{-1}} = a^2 i$. The quantity a^2 attains all positive real values, so that the H_1-orbit of i is $i\mathbb{R}_{>0}$, an open vertical half-line in $\mathbb{H}$ starting at 0.
- Let $M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in H_2$. Then $M \cdot i = i + t$. Hence the H_2-orbit of i is the horizontal line in $\mathbb{C}$ passing through i.
- Let $M = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in H_3$. Then

$$M \cdot i = \frac{i}{ti+1} = \frac{t}{1+t^2} + i\frac{1}{1+t^2}.$$

Let $x = \Re(M \cdot i) = \frac{t}{1+t^2}$ and $y = \Im(M \cdot i)$. Then we see that (x, y) can be any pair of values such that $0 < y < 1$ and

$$x^2 + y^2 = y.$$

This is the equation of the circle in the complex plain with center in $\frac{i}{2}$ and radius 1/2.

5. Let G be a finite group and $H \subset G$ a subgroup. Suppose that the index of H in G is equal to the smallest prime number dividing $|G|$. Prove: $H \lhd G$. [*Hint:* Define a suitable action $\rho : G \longrightarrow \mathrm{Sym}(G/H)$. Look at $\ker(\rho)$ and $\mathrm{Card}(\mathrm{Im}(\rho))$.]

Solution: Let G be a finite group, p the minimal prime dividing $|G|$ and $H \leqslant G$ subgroup of index $[G : H] = p$. The group G acts on G/H by left multiplication, giving a group homomorphism

$$\rho : G \longrightarrow \mathrm{Sym}(G/H).$$

The kernel is

$$\begin{aligned}\ker(\rho) &= \{g \in G : \forall x \in G, gxH = xH\} = \{g \in G : \forall x \in G, x^{-1}gx \in H\} = \qquad (1)\\ &= \{g \in G : \forall x \in G, g \in xHx^{-1}\} = \bigcap_{x \in G} xHx^{-1} \subseteq H. \qquad (2)\end{aligned}$$

By Lagrange's Theorem,

$$\mathrm{Card}(\mathrm{Im}(G)) \,|\, \mathrm{Card}(\mathrm{Sym}(G/H)) = [G : H]! = p!.$$

By the First Isomorphism Theorem, $\mathrm{Im}(\rho) \cong G/\ker(\rho)$, so that

$$\mathrm{Card}(\mathrm{Im}(\rho)) \,|\, \mathrm{Card}(G).$$

Since the prime factors of $\mathrm{Card}(G)$ are all bigger or equal to p, while the prime factors of $p!$ are all smaller or equal to p (which itself has exponent 1 in the prime decomposition of $p!$, we deduce that $\mathrm{Card}(\mathrm{Im}(G)) \in \{1, p\}$. Then

$$\{1, p\} \ni \mathrm{Card}(\mathrm{Im}(G)) = [G : \ker(\rho)] \overset{(1)}{\geqslant} [G : H] = p$$

implies that $[G : H] = [G : \ker(\rho)]$, so that $H = \ker(\rho)$ again by (1). Since kernels are normal subgroup, we can conclude that $H \lhd G$.

6. Let G be a finite group and p a prime number. Let $\mathcal{T}_p$ be the set of all p-Sylow subgroups and fix $P \in \mathcal{T}_p$. Since conjugation preserves cardinality of subsets, G acts on $\mathcal{T}_p$ by

$$g \cdot H = gHg^{-1}.$$

   (a) Show that the induced action of P on $\mathcal{T}_p$ has a unique fixed point.
   (b) Deduce that $\mathrm{Card}(\mathcal{T}_p) \equiv 1 \pmod{p}$.
   (c) Prove that $\mathrm{Card}(\mathcal{T}_p) \,|\, m := [G : P]$. [*Hint:* Use the action of G by conjugation on the set of its subgroups]
   (d) Let $M \supset P$ be a subgroup of G containing $N_G(P)$. Prove that $N_G(M) = M$.

Solution:

   (a) Let $H \in \mathcal{T}_p$ be fixed by P. Then $xHx^{-1} = H$ for each $x \in P$. This means that P is a subgroup of $N_G(H)$. Since $\mathrm{Card}(N_G(H))|\mathrm{Card}(G)$, both H and P are p-Sylow subgroups of $N_G(H)$, so that they are conjugates in $N_G(H)$. But H is stable under conjugation by elements in $N_G(H)$ (i.e., $H \lhd N_G(H)$) by definition of $N_G(H)$, so that $H = P$. On the other hand $P \in \mathcal{T}_p$ is clearly fixed by P. This implies that P is the only point in $\mathcal{T}_p$ which is fixed by P.

(b) Let $H \in \mathcal{T}_p \smallsetminus \{P\}$. Denote by $\mathrm{orb}_P(H)$ the orbit of H under the action of P on $\mathcal{T}_p$ by conjugation. By part (a), $\mathrm{Card}(\mathrm{orb}_P(H)) > 1$. By the orbit-stabilizer Theorem,

$$\mathrm{Card}(\mathrm{orb}_P(H)) = [P : \mathrm{Stab}_P(H)]$$

so that $\mathrm{Card}(\mathrm{orb}_P(H)) \neq 1$ divides a power of p, meaning that it is divisible by p. We can than conclude that

$$\mathrm{Card}(\mathcal{T}_p) = \sum_{P\text{-orbits } U \subset \mathcal{T}_p} \mathrm{Card}(U) = 1 + \sum_{P\text{-orbits } \{P\} \neq U \subset \mathcal{T}_p} \mathrm{Card}(U) \in 1 + p\mathbb{Z}$$

as desired.

(c) Since all p-Sylow subgroups are conjugated in G and all conjugates of p-Sylow subgroups are p-Sylow subgroups, the number of p-Sylow subgroups is equal to the cardinality of the G-orbit of P with respect to the action of G on the set of its subgroups by conjugation. Notice that $\mathrm{Stab}_G(P) = N_G(P)$ by definition. By the orbit stabilizer theorem, we can conclude:

$$\mathrm{Card}(\mathcal{T}_p) = [G : N_G(P)] \mid [G : P]$$

(d) It is clear that $M \subset N_G(M)$, so we now prove that $N_G(M) \subset M$. Let $x \in N_G(M)$. Since $P \subset M$,

$$xPx^{-1} \subset xMx^{-1} = M.$$

This means that P and xPx^{-1} are both Sylow subgroups of M, meaning that there exists $y \in M$ such that $yPy^{-1} = xPx^{-1}$, i.e., $P = (y^{-1}x)P(y^{-1}x)^{-1}$. Hence $y^{-1}x \in N_G(P) \subset M$, implying that $x = y(y^{-1}x) \in M$.

7. Let K be a field and D be the subgroup of $G := \mathrm{GL}_2(K)$ consisting of diagonal matrices. Determine $N_G(D)$ and $N_G(D)/D$.

Solution: If $K = \mathbb{F}_2$, then $D = \{\mathrm{Id}_2\}$, so that $N_G(D)/D \cong N_G(D) = \mathrm{GL}_2(K)$.

If $K \neq \mathbb{F}_2$, then D contains both scalar matrices and matrices with different two distinct entries in the diagonal. Since scalar matrices commute with all other matrices, we deduce that

$$N_G(D) = \{g \in \mathrm{GL}_2(K) : \forall \lambda, \mu \in K \text{ s.t. } \lambda \neq \mu, g \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} g^{-1} \in D\}.$$

If the matrix $g \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} g^{-1}$ above is diagonal, then it can be either $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ or $\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$, because similar matrices have the same eigenvalues. Thus we have two sufficient conditions for g in the above expression of $N_G(D)$, which we parse for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- First possibility: $g\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} g$. This equality reads $\begin{pmatrix} a\lambda & b\mu \\ c\lambda & d\mu \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\mu & d\mu \end{pmatrix}$, which holds if and only if $b = c = 0$, because $\lambda \neq \mu$. This possibility is equivalent to ask that $g \in D$.
- Second possibility: $g\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} g$. This equality reads $\begin{pmatrix} a\lambda & b\mu \\ c\lambda & d\mu \end{pmatrix} = \begin{pmatrix} a\mu & b\mu \\ c\lambda & d\lambda \end{pmatrix}$, which implies that $a = d = 0$. This possibility is equivalent to ask that $g \in \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D$.

This proves:

$$N_G(D) := D \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right\}.$$

This tells us that $[N_G(D) : D] = 2$, so that $N_G(D)/D$ has cardinality 2 and as such is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

8. Let S_n act on $\{1, \ldots, n\}$. Define an action of S_n on $\{1, \ldots, n\} \times \{1, \ldots, n\}$ by $g \cdot (i, j) = (g(i), g(j))$. Show that this action has exactly two orbits and determine them.

   Solution: For $\sigma \in S_n$ and $i, j \in \{1, \ldots, n\}$, we know that $\sigma(i) = \sigma(j)$ if and only if $i = j$. This means that an element $(k, k) \in \{1, \ldots, n\}^2$ cannot lie in the same orbit of an element $(i, j) \in \{1, \ldots, n\}^2$ with $i \neq j$. Hence we have at least two orbits. In order to conclude, we need to check that all elements (k, k) are in the same orbit and all elements (i, j) for $i \neq j$ are in the same orbit.

   - Let $k, k' \in \{1, \ldots, n\}$. As seen in class, the action on S_n is transitive, so that there is $\sigma \in S_n$ such that $\sigma(k) = k'$, implying that $\sigma \cdot (k, k) = (k', k')$.
   - Let, $i, j, i', j' \in \{1, \ldots, n\}$ be elements such that $i \neq j$ and $i' \neq j'$. Then the sets $\{1, \ldots, n\} \smallsetminus \{i, j\}$ and $\{1, \ldots, n\} \smallsetminus \{i', j'\}$ have the same cardinality so that there is a bijective map $\{1, \ldots, n\} \smallsetminus \{i, j\} \longrightarrow \{1, \ldots, n\} \smallsetminus \{i', j'\}$. We extend this to a bijection $\sigma \in S_n$ by sending $i \mapsto i'$ and $j \mapsto j'$. Then

   $$\sigma(i, j) = (i', j').$$

   This concludes the proof that there are two orbits: $\{(k, k) \in \{1, \ldots, n\}^2\}$ and $\{(i, j) \in \{1, \ldots, n\}^2 : i \neq j\}$.