1. Does there exist a nontrivial group action of the group $\mathbb{Z}/7\mathbb{Z}$
   (a) on a set with 6 elements?
   (b) on a set with 8 elements?
2. Let $\mathbb{F}_p$ be the field $\mathbb{Z}/p\mathbb{Z}$.
   (a) How many monic polynomials of degree 2 with coefficients in $\mathbb{F}_p$ are irreducible over $\mathbb{F}_p$?
   (b) How many monic polynomials of degree 3 with coefficients in $\mathbb{F}_p$ are irreducible over $\mathbb{F}_p$?
3. Let $G$ be a group of order 12.
   (a) Must $G$ be abelian?
   (b) List all possible abelian $G$ of order 12 (up to isomorphism).
4. Is the ideal generated by $x^7 + 1$ and $x^5 + 1$ a principal ideal in $\mathbb{F}_2[x]$? If not, why? If so, find a generator.
5. How many elements in $S_5$ have order exactly 6?
6. Let $d$ be the smallest index of a proper subgroup of $S_8$. What is $d$? Do there exist two distinct subgroups of $S_8$ of index $d$?
7. Let $G$ be a finite group. Let $H$ be a subgroup of $G$ which is not normal. Can the index of $H$ be 2? Can the index of $H$ be 3?
8. Are the following rings UFDs?
   (a) $\mathbb{Z}[x]$
   (b) $(\mathbb{Z}/2\mathbb{Z})[[x]]$
   (c) $\mathbb{R}[x]/(x^2 - 1)$
   (d) $\mathbb{R}[x]/(x^2 + 1)$

   Explain why or why not.
9. Which of the following rings are fields?
   (a) $\mathbb{Z}[x]/(3, x^2 + x + 1)$
   (b) $\mathbb{Z}[x]/(4, x^2 + x + 1)$
   (c) $\mathbb{Z}[x]/(2, x^2 + x + 1)$

   Explain why or why not.

10. What is the automorphism group $\mathrm{Aut}(G)$ of the following groups $G$?
    (a) $\mathbb{Z}/2\mathbb{Z}$
    (b) $\mathbb{Z}/8\mathbb{Z}$
    (c) $\mathbb{Z}$
    (d) $S_3$

11. In the ring $\mathbb{C}[x, y]$, do the following inclusions of ideals hold?
    (a) $(x-1, y-2) \subset (y^2 - 4x)$
    (b) $(y^2 - 4x) \subset (x-1, y-2)$

    Prove your answer.

12. Let $a \in \mathbb{Z}$ be a number which satisfies

$$\bar{a} = \bar{3} \bmod 4,$$
$$\bar{a} = \bar{1} \bmod 3.$$

    Does this information suffice to compute $\bar{a}$ mod 12? If so, what is the answer?

13. List all subgroups of $S_3$. Which of those subgroups are normal?

14. Describe all possible homomorphisms
    (a) $f : S_3 \to \mathbb{Z}/6\mathbb{Z}$
    (b) $g : \mathbb{Z}/6\mathbb{Z} \to S_3$

15. Which of the following statements are true for all groups $G$ of order 56?
    (a) They contain a subgroup of order 2.
    (b) They contain a subgroup of order 3.
    (c) The number of different subgroups of order 8 contained in $G$ is exactly 2.

16. In the ring $R = \mathbb{R}[x]$, are the following subsets ideals of $R$? Prove your answer.
    (a) $\{f(x) \in R : f(\pi) = 0\}$
    (b) $\{f(x) \in R : f'(\pi) = 0\}$
    (c) $\{f(x) \in R : f(i) = 0\}$
    (*Note*: we can evaluate a real polynomial at the complex number $i \in \mathbb{C}$)
    (d) $\{f(x) \in R : f(0) = f(1)\}$

17. Compute the conjugate of the group element $a = (3\,5\,2\,4) \in S_5$ by the element $(2\,3)$. How many elements of $S_5$ are conjugate to $a$?

18. Let $\mathbb{F}_2$ be the field with two elements.

    (a) Show that, for every degree $n \geq 1$, there exists an irreducible polynomial of degree $n$ in $\mathbb{F}_2[x]$.

    (b) Let $\overline{\mathbb{F}}_2$ be the algebraic closure of $\mathbb{F}_2$. Show that $\overline{\mathbb{F}}_2$ is infinite.

19. Let $A$ be a module over the ring $\mathbb{C}[x]$. Since $\mathbb{C} \subset \mathbb{C}[x]$, $A$ is a $\mathbb{C}$-vector space.

    (a) List all $\mathbb{C}[x]$-modules $A$ with $\dim_{\mathbb{C}} A = 1$, up to isomorphism.

    (b) List all $\mathbb{C}[x]$-modules $A$ with $\dim_{\mathbb{C}} A = 2$, up to isomorphism.

20. Does the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2$ have degree 3 over $\mathbb{Q}$?