Formulary

Theorem 2.2.1 (Cauchy-Riemann Equations)**.** Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{R}^2 = \mathbb{C}$ real-differentiable at $z_0 \in \Omega$. Then $f = u + iv$ is complex-differentiable at z_0 if and only if

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Theorem 3.1.1. Let $\Omega \subset \mathbb{C}$ be open. For $f \in C^0(\Omega; \mathbb{C})$, the following are equivalent:

i) $\exists F \in C^1(\Omega; \mathbb{C})\colon F$ holomorphic, $F' = f$;

ii) for every $\gamma \in C^1_{\text{pw}}([0,1]; \Omega)$, the integral $\int_\gamma f(z)\,\mathrm{d}z$ depends only on $\gamma(0)$ and $\gamma(1)$;

iii) $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed path $\gamma \in C^1_{\text{pw}}([0,1]; \Omega)$.

Corollary 3.3.1 (Cauchy)**.** Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ analytic. Let $U \subset \bar{U} \subset \Omega$ be bounded and $\partial U \in C^1_{\text{pw}}$. Then

$$\int_{\partial U} f(z)\,\mathrm{d}z = 0.$$

Theorem 3.4.1 (Cauchy's Integral Formula)**.** Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ analytic, and let $B \subset \overline{B} \subset \Omega$ be a disc. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \quad \text{for all } z \in B.$$

Corollary 3.4.2. Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ analytic. Then f is smooth; in particular, f is holomorphic. More precisely:

i) f is infinitely differentiable in the interior of every ball $B := B_R(z_0) \subset \overline{B} \subset \Omega$ with
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta \quad \text{for all } z \in B \text{ and } n \in \mathbb{N}_0.$$

ii) f is represented by its Taylor series
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$
locally around every point $z_0 \in \Omega$.

 Please turn the page.

Remark 3.4.1 (Cauchy's Inequalities)**.** Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ analytic. For every $z_0 \in \Omega$ and every $R > 0$ with $\overline{B_R(z_0)} \subset \Omega$ we have

$$|f^{(n)}(z_0)| \leqslant \frac{n!\max_{\partial B_R(z_0)} |f|}{R^n} \quad \text{for all } n \in \mathbb{N}.$$

Corollary 3.4.6 (Weierstrass)**.** Let $\Omega \subset \mathbb{C}$ be open and $(f_k)_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{C})$ a sequence of holomorphic functions with $f_k \xrightarrow{k\to\infty} f$ in $C^0_{\text{loc}}(\Omega,\mathbb{C})$; i.e.,

$$\sup_{z\in K} |f_k(z) - f(z)| \to 0 \quad (k \to \infty) \quad \text{for every compact } K \subset \Omega.$$

Then f is holomorphic and $f_k^{(n)} \xrightarrow{k\to\infty} f^{(n)}$ in $C^0_{\text{loc}}(\Omega;\mathbb{C})$ for every $n \in \mathbb{N}$.

Corollary 3.4.7 (Maximum Principle)**.** Let $\Omega \subset \mathbb{C}$ be open, bounded and connected, and let $f \in C^0(\overline{\Omega};\mathbb{C})$ be analytic in Ω. Then

$$M := \max_{z\in\overline{\Omega}} |f(z)| = \max_{z\in\partial\Omega} |f(z)|;$$

and if $|f(z_0)| = M$ for some $z_0 \in \Omega$, then $f \equiv f(z_0)$.

Theorem 3.4.3 (Schwarz Lemma)**.** Let $B = B_1(0)$ and let $f\colon B \to \mathbb{C}$ be analytic with $f(0) = 0$ and $|f(z)| \leqslant 1$ for all $z \in B$. Then

$$|f'(0)| \leqslant 1 \quad \text{and} \quad |f(z)| \leqslant |z| \quad \text{for all } z \in B.$$

If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in B$, then $f(z) = cz$ is linear for some $c \in \mathbb{C}$ with $|c| = 1$.

Theorem 4.1.2 (Residue Theorem)**.** Let $\Omega \subset \mathbb{C}$ be open, $Z := \{z_1, ..., z_K\} \subset \Omega$ and $f\colon \Omega\setminus Z \to \mathbb{C}$ holomorphic with poles at $z_k, 1 \leqslant k \leqslant K$. Then for every disc $B := B_R(z_0)$ with $R > 0$ such that $\overline{B_R(z_0)} \subset \Omega$ and $\partial B_R(z_0) \subset \Omega \setminus Z$ we have

$$\frac{1}{2\pi i}\int_{\partial B_R(z_0)} f(z)\,\mathrm{d}z \;=\; \sum_{z_k\in B_R(z_0)} \operatorname{Res}_{z_k} f.$$

Theorem 4.3.1 (Argument Principle)**.** Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ meromorphic. Let U be a bounded set with $\overline{U} \subset \Omega$ such that f has neither poles, nor zeros on ∂U. Then

$$\frac{1}{2\pi i}\int_{\partial U} \frac{f'(z)}{f(z)}\,\mathrm{d}z \;=\; \sum_{z_k\in U} n_k - \sum_{z_j\in U} m_j,$$

where z_k runs over the zeros of f of order $n_k \in \mathbb{N}$ and z_j over the poles of f of order $m_j \in \mathbb{N}$.

Theorem 4.3.2 (Rouché)**.** Let $\Omega \subset \mathbb{C}$ be open and $f, g\colon \Omega \to \mathbb{C}$ holomorphic. Suppose that for $B := B_R(z_0)$ with $\bar{B} \subset \Omega$ we have the inequality

$$|f(z)| > |g(z)| \quad \text{for all } z \in \partial B.$$

Then f and $f + g$ have the same number of zeros (counted with multiplicity) in B.

Theorem 5.3.1 (Jensen)**.** Let $\Omega \subset \mathbb{C}$ be open, $B := B_R(0) \subset \bar{B} \subset \Omega$ and $f\colon \Omega \to \mathbb{C}$ holomorphic with $f(z) \neq 0$ for $z \in \partial B$ and $f(0) \neq 0$. Then

$$\log|f(0)| = \sum_{k=1}^{K} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,\mathrm{d}\theta,$$

where $z_1, ..., z_K$ are the zeros of f in B and every zero appears in the list n_k times according to its multiplicity.

Theorem 5.3.4 (Phragmén-Lindelöf)**.** Let $S := \{z = re^{i\theta} \in \mathbb{C} \mid r > 0, |\theta| < \frac{\pi}{4}\}$, and let $f \in C^0(\bar{S}; \mathbb{C})$ be holomorphic on S with growth order $\rho_f \leq 1$ and

$$|f(z)| \leq 1 \quad \text{for all } z \in \partial S.$$

Then $|f(z)| \leqslant 1$ for every $z \in S$.

Theorem 5.3.5. Let $\Omega \subset \mathbb{C}$ be open and $f_n\colon \Omega \to \mathbb{C}$, $n \in \mathbb{N}$ holomorphic with

$$|f_n(z) - 1| \leq c_n \quad \text{for all } z \in \Omega,$$

for certain real $c_n > 0$ such that $\sum_{n=1}^{\infty} c_n < \infty$. Then the product $F(z) := \prod_{n=1}^{\infty} f_n(z)$ converges independent of the order of the factors, and $F\colon \Omega \to \mathbb{C}$ is holomorphic.
Moreover, for $z \in \Omega$ with $f_n(z) \neq 0$ for all $n \in \mathbb{N}$ we have

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)}.$$

Sheet 7, Exercise 4(a). Let $f(z)$ be a rational function with no poles on the real axis. If the function $z \mapsto f(\frac{1}{z})$ has a zero of order at least two at $z = 0$, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\substack{a \text{ Polstelle} \\ \mathrm{Im}(a)>0}} \mathrm{Res}_a\, f.$$

Sheet 7, Exercise 5(a). Let $f(z)$ be a rational function with no poles on the real axis. If the function $z \mapsto f(\frac{1}{z})$ has a zero at $z = 0$, then

$$\int_{-\infty}^{\infty} f(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{\substack{a \text{ Polstelle} \\ \mathrm{Im}(a)>0}} \mathrm{Res}_a(f(z)e^{iz}).$$

Sheet 8, Exercise 4. The biholomorphic maps $\mathbb{C} \to \mathbb{C}$ are precisely those of the form

$$f\colon \mathbb{C} \to \mathbb{C},\ z \mapsto az + b \quad \text{with } a, b \in \mathbb{C},\ a \neq 0.$$

Sheet 10, Exercise 2. Let $\Omega \subset \mathbb{C}$ be open and connected, and let $f\colon \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic. There exists a branch of $\log f$ on Ω if and only if

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 0$$

for every closed path $\gamma \in C^1_{\mathrm{pw}}([0,1], \Omega)$.