Problem set 3

Definition: $I := [0, 1]$. For a space X, we define its *suspension* SX as the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. For a map $f : X \to Y$, its *suspension* $Sf : SX \to SY$ is defined as the quotient map of $f \times id_I : X \times I \to Y \times I$.
Remark: If $X = S^n$, then $SX \approx S^{n+1}$.

1. (a) If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.
   (b) Let $f : S^n \to S^n$ be a map of degree 0. Show that there exist points $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$.
2. Let G be a group acting freely on S^n. Prove that if n is even, then G is either trivial or isomorphic to $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.
3. We view S^1 as the unit circle in $\mathbb{C}$. Show that for each $k \in \mathbb{Z}$ the map $f : S^1 \to S^1$, given by $f(z) = z^k$, has degree k.
4. (a) Let $Sf : S^{n+1} \to S^{n+1}$ be the suspension of a map $f : S^n \to S^n$. Show that $\deg Sf = \deg f$.
   (b) Show that for each $n \geq 1$ and each $k \in \mathbb{Z}$ there is a map $f : S^n \to S^n$ of degree k.
5. Construct a surjective map $S^n \to S^n$ of degree 0 for each $n \geq 1$.
   (*Hint:* Do it first for $n = 1$ and then use exercise 4a).)
6. Show that $SO(n)$ is path connected.