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Solutions to problem set 2

1. Sketch of a short solution:

   Use the long exact sequence of the triple (X, A, pt) and the fact that $H_*(X, pt) \cong \widetilde{H}_*(X)$. Alternatively, one can use the long exact sequence of reduced homology groups for the pair (X, A) $(A \neq \emptyset)$ and conclude by using that $H_*(X, A) = \widetilde{H}_*(X, A)$. (See p. 118 in Hatcher's book.)

   The following is a more detailed solution:

   Consider the long exact sequence for the pair (X, A):

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to \ldots \tag{1}$$

   Since $H_p(A) = \widetilde{H}_p(A)$ for $p > 0$ and $\widetilde{H}_*(A) = 0$ by assumption, this long exact sequence yields exact sequences

$$0 \to H_p(X) \to H_p(X, A) \to 0 \quad \text{for } p \geq 2,$$

   which tells us that

$$\widetilde{H}_p(X) \cong H_p(X, A) \quad \text{for } p \geq 2.$$

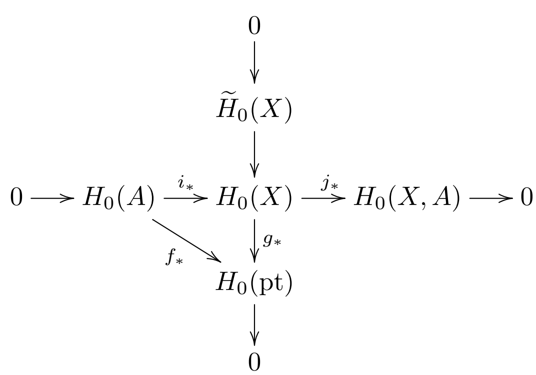
   The "right end" of (1) is

$$0 \to H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \to 0. \tag{2}$$

   It follows straight from the definitions that the composite $H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{f_*} H_0(\mathrm{pt})$ vanishes, where f_* is induced by $f : A \to \mathrm{pt}$. Therefore $\operatorname{im} \partial_* \subseteq \ker f_* = \widetilde{H}_0(A)$; since the latter is zero by assumption, it follows that $\partial_* = 0$. This implies, first, that $j_* : H_1(X) \to H_1(X, A)$ is also an isomorphism, and hence

$$\widetilde{H}_1(X) \cong H_1(X, A).$$

   Second, $\partial_* = 0$ implies that we get from (2) the horizontal short exact sequence in



   The vertical sequence (in which g_* is induced by $g : X \to \mathrm{pt}$, and where the first map is the inclusion of $\widetilde{H}_0(X) = \ker g_*$) is also exact. Moreover, the lower left triangle commutes, and thus $i_* \circ f_*^{-1}$ is a right-inverse of g_* (note that f_* is an isomorphism because $\ker f_* = \widetilde{H}_0(A) = 0$ by assumption). Hence the vertical sequence splits. Combining these facts, we obtain

$$\widetilde{H}_0(X) \cong H_0(X)/H_0(\mathrm{pt}) \cong H_0(X)/H_0(A) \cong H_0(X, A).$$

2. Recall that we may view singular 0-chains in X as finite formal sums $\sum_x n_x x$ with $x \in X$ and $n_x \in \mathbb{Z}$. In particular, a zero-simplex in X is a point $x \in X$.

   By definition, the image of $[x] \in H_0(X)$ under $f_* : H_0(X) \to H_0(X)$ is the class of $f(x) \in X$, viewed as a 0-simplex. Since X is path-connected, we can choose a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = f(x)$; regarding this path as a 1-simplex $\gamma : \Delta_1 \to X$, we obtain

$$\partial_1 \gamma = \gamma(1) - \gamma(0) = f(x) - x.$$

   Hence the 0-chain $f(x) - x$ is a boundary, and thus $f_*[x] - [x] = [f(x) - x] = 0 \in H_0(X)$.

3. Let $\gamma : I \to X$ be a loop based at x_0, and recall that we can also consider γ as a singular 1-cycle; we denote the corresponding classes by $[\gamma] \in \pi_1(X, x_0)$ and $[\![\gamma]\!] \in H_1(X)$. It follows straight from the definitions of $f_\sharp, f_*$ and the Hurewicz homomorphisms ϕ_X and ϕ_Y that

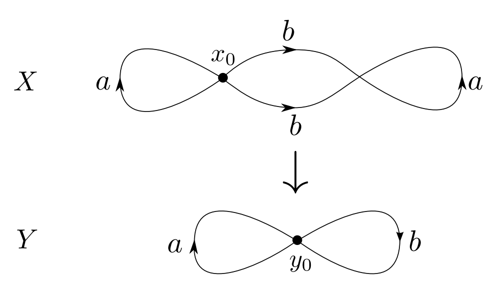
$$f_*(\phi_X([\gamma])) = f_*[\![\gamma]\!] = [\![f \circ \gamma]\!] = \phi_Y([f \circ \gamma]) = \phi_Y(f_\sharp[\gamma]).$$

   Since this works for every γ, we conclude $f_* \circ \phi_X = \phi_Y \circ f_\sharp$.

4. Denote by $p_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ the map induced by p. Let $\gamma : I \to X$ be a loop based at x_0 and suppose that $p_\sharp([\gamma]) = 0 \in \pi_1(Y, y_0)$, which is equivalent to saying that the loop $p \circ \gamma : I \to Y$ is null-homotopic. This means that there exists a homotopy $F : I \times I \to Y$ such that $F(\cdot, 0) = p \circ \gamma$ and $F(\cdot, 1) \equiv y_0$ is constant. Since γ lifts $F(\cdot, 0)$, the Covering Homotopy Theorem tells us that there is a (unique) homotopy $G : I \times I \to X$ such that $G(\cdot, 0) = \gamma$ and such that $p \circ G = F$. In particular, this implies that $p(G(\cdot, 1)) = F(\cdot, 1)$ is constant, and thus that $G(\cdot, 1)$ is constant, because p is a covering map and hence a local homeomorphism. It follows that $[\gamma] = 0 \in \pi_1(X, x_0)$, and thus $p_\sharp$ is a monomorphism.

   It is not true that $p_* : H_1(X) \to H_1(Y)$ needs to be a monomorphism. For example, take any space Y with $H_1(Y) \neq 0$, set $X = Y \sqcup Y$, and consider the obvious double cover $p : X \to Y$; the induced map $p_* : H_1(X) \cong H_1(Y) \oplus H_1(Y) \to H_1(Y)$, $(\alpha, \beta) \mapsto \alpha + \beta$, is clearly not injective.

   For a slightly more involved example, consider $X = S^1 \vee S^1 \vee S^1$, $Y = S^1 \vee S^1$ and the covering map $p : X \to Y$ indicated by the following picture (convince yourself that this is a covering map!):

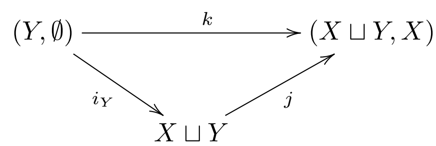


   Consider now the loop γ in X that starts at x_0 and then winds once around all of X in clockwise direction. This loop defines a non-zero element $[\![\gamma]\!] \in H_1(X)$; but note that

$$p_*[\![\gamma]\!] = \phi_Y(p_\sharp[\gamma]) = \phi_Y[b^{-1}a^{-1}ba] = 0 \in H_1(Y),$$

   because $[b^{-1}a^{-1}ba]$ lies in the commutator of $\pi_1(Y, y_0)$, which is the kernel of the Hurewicz homomorphism ϕ_Y. Thus $p_* : H_1(X) \to H_1(Y)$ is not a monomorphism.

5. The commutativity of the diagram



implies that

$$k_* = j_* \circ (i_Y)_* : H_*(Y) \to H_*(X \sqcup Y, X) \tag{3}$$

by functoriality of H. Since k_* is an isomorphism by the Excision axiom, it follows that j_* is surjective. This implies that the connecting homomorphisms ∂_* in the long exact sequence for the pair $(X \sqcup Y, X)$ are all zero, and hence this long exact sequence gives rise to short exact sequences

$$0 \to H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \to 0. \tag{4}$$

Now equation (3) is equivalent to $j_* \circ (i_Y)_* \circ k_*^{-1} = \mathrm{id}_{H_p(X \sqcup Y, X)}$, and thus j_* has a right inverse. Hence the short exact sequence (4) splits, and therefore

$$(i_X)_* \oplus ((i_Y)_* \circ k_*^{-1}) : H_p(X) \oplus H_p(X \sqcup Y, X) \to H_p(X \sqcup Y)$$

is an isomorphism; precomposing it with the isomorphism $\mathrm{id}_{H_p(X)} \oplus k_*$ yields the desired isomorphism

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \to H_p(X \sqcup Y).$$