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Algebraic Topology I

HS 2018

Solutions to problem set 3

1. (a) Since $f(x) \neq x$, $\forall x \in S^n$, the line segment $(1-t)f(x) - tx$, $t \in [0,1]$, does not pass through 0. Therefore, if f has no fixed points,

$$f_t(x) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

is a well defined homotopy from f to the antipodal map $-id$ which has degree $\deg(-id) = (-1)^{n+1}$. Thus $\deg f = (-1)^{n+1}$.

(b) Since $\deg f = 0 \neq (-1)^{n+1}$ it must have a fixed point $x \in S^n$ by exercise 1a), i.e. $f(x) = x$. For the same reason, since $g := (-id) \circ f$ has degree $\deg g = \deg(-id) \cdot \deg f = 0$ there exists a fixed point $y \in S^n$ of g, i.e. $g(y) = -f(y) = y$. Which means that $f(y) = -y$.

2. Let the group action be given by the homomorphism $\rho : G \to Homeo(S^n)$. The degree of a homoemorphism is always ± 1. Therefore the group action determines a degree function $d : G \to \{\pm 1\}$ given by $d(g) := \deg \rho(g)$. Furthermore d is a homomorhism:

$$d(hg) = \deg \rho(hg) = \deg(\rho(h) \circ \rho(g)) = \deg \rho(h) \cdot \deg \rho(g) = d(h) \cdot d(g).$$

If $g \in G$ is a non trivial element, then $\rho(g)$ has no fixed points as the action is free and hence (by exercise 1a)) we have $d(g) = (-1)^{n+1}$. So, if n is even, the kernel of d is trivial which implies that G is isomorphic to a subgroup of $\{\pm 1\} \cong \mathbb{Z}_2$.

3. See example 2.32 on page 137 in Hatcher's book.

4. (a) Let $I := [0,1]$ and let $CS^n := (S^n \times I)/(S^n \times \{1\})$ denote the cone with base $S^n = S^n \times \{0\} \subset CS^n$. Notice that CS^n/S^n is the suspension of S^n and in particular $CS^n/S^n \cong S^{n+1}$. The map f induces $Cf : (CS^n, S^n) \to (CS^n, S^n)$ with quotient Sf. Since $S^n \times \{0\}$ is a deformation retract of some open neighborhood in CS^n we get (see Proposition 2.22 in Hatcher) that

$$H_{n+1}(CS^n, S^n) \cong H_{n+1}(CS^n/S^n, S^n/S^n) \cong \widetilde{H}_{n+1}(CS^n/S^n) \cong \widetilde{H}_{n+1}(S^{n+1}).$$

The naturality of the boundary maps in the long exact sequence of the pair (CS^n, S^n) implies that we have the following commutative diagram

$$\begin{CD}
\widetilde{H}_{n+1}(S^{n+1}) @>{\partial_*}>{\cong}> \widetilde{H}_n(S^n) \\
@V{Sf_*}VV @VV{f_*}V \\
\widetilde{H}_{n+1}(S^{n+1}) @>{\partial_*}>{\cong}> \widetilde{H}_n(S^n)
\end{CD}$$

The two horizontal maps are isomorphisms since CS^n is contractible and hence $\widetilde{H}_*(CS^n) = 0$ in the long exact sequence of the pair. Therefore, if f_* is multiplication by $d = \deg f$, then Sf_* is also multiplication by d and hence $\deg f = \deg Sf$.

(b) Given $k \in \mathbb{Z}$ the map $S^1 \to S^1 : z \mapsto z^k$ has degree k. Now assume that we have constructed a map $f : S^n \to S^n$ of degree k, then (by exercise 4a)), the map $Sf : S^{n+1} \to S^{n+1}$ has degree k as well. So, the claim follows by induction.

5. First, let $n = 1$ and denote $I := [0,1]$. Let $g : I \to \mathbb{R}$ be a continuous map such that $g(0) = g(1) = 0$ and $g(1/2) = 2\pi$. The map g induces a well defined continuous surjection $f : I/\partial I = S^1 \to S^1 : t \mapsto e^{ig(t)}$. By the path lifting property the map g is the unique lift of f to the universal cover $\mathbb{R}$ of S^1 starting at the point $0 \in \mathbb{R}$. So $f \in p_\sharp \underbrace{\pi_1(\mathbb{R}, 0)}_{=0} \subset \pi_1(S^1, 1)$ is homotopic to a constant map (which is clearly not surjective an therefore has degree 0) and hence $\deg f = 0$. Here, $p : \mathbb{R} \to S^1$ is the universal cover.

   Using exercise 4a) we obtain, by repeatedly suspending the map f, a surjective map $S^n \to S^n$ of degree 0.

   For an alternative, more explicit, solution see example 2.31 in Hatcher's book.

6. For $n = 2$ we have that $SO(2)$ is homeomorphic to the circle S^1 which is path connected. Proceeding by induction we assume that $SO(n-1)$ is path connected. Given any $A \in SO(n)$ it is enough to show that there is a path in $SO(n)$ connecting A to the identity matrix I_n. This means that we need to find a continuous path taking the standard basis $e_1, \dots, e_n$ to their image $Ae_1, \dots, Ae_n$. Let $\Lambda \subset \mathbb{R}^n$ be a plane containig both e_1 and Ae_1. By the path connectedness of $SO(2)$, we can continuously move e_1 to Ae_1 by a rotation R of the plane Λ.

   It remains to continuously move $Re_2, \dots, Re_n$ to $Ae_2, \dots, Ae_n$ while keeping Ae_1 fixed. Notice that $Ae_1 = Re_1 \perp Re_i$ and $Ae_1 \perp Ae_i$ for each $2 \leq i \leq n$ since both R and A preserve angles. Hence the required motion can take place in the hyperplane $\mathbb{R}^{n-1}$ of vectors orthogonal to Ae_1, where it exists by the assumption that $SO(n-1)$ is path connected.

   Concatenating the two motions gives a path in $SO(n)$ from I_n to A and thus $SO(n)$ is path connected.