The aim of this exercise sheet is to recall the main definitions and properties of representations.

**Definition 1.** Let $G$ be a group and $k$ be a field. A linear representations of $G$, defined over $k$, is a group homomorphism

$$
\rho : G \to \text{GL}(E),
$$

where $E$ is a $k$-vector space. The dimension of $E$ is called the dimension of $\rho$, or sometimes its rank, or degree. We will denote it $\text{dim}(\rho)$. If $\rho$ is a finite dimensional representation, we define the character associated to $\rho$,

$$
\chi_\rho : G \to k \quad g \mapsto \text{Tr}(\rho(g))
$$

**Definition 2.** Let $G$ be a group and $\rho : G \to \text{GL}(E)$ a $k$-representation of $G$

i) We say that a subspace $F \subseteq E$, is said to be stable under $\rho$ if $\rho(g) \cdot F \subseteq F$ for any $g \in G$. Then the $k$-representation

$$
\rho_F : G \to \text{GL}(F) \quad g \mapsto \rho(g)|_F,
$$

is called subrepresentation of $\rho$.

ii) We say that $\rho$ is an irreducible representation if $E \neq 0$ and there is no stable subspace of $E$ under $\rho$, except 0 and $E$ itself.

iii) We say that $\rho$ is semisimple if it can be written as a direct sum of subrepresentation, each of which is irreducible:

$$
\rho \cong \bigoplus_{i \in I} \rho_i.
$$

**Definition 3.** Let $G$ be a group and $k$ a field. A morphism, or homomorphism, between two $k$-representation $\rho_1$ and $\rho_2$ of $G$, acting on the vector spaces $E_1$ and $E_2$, respectively, is a linear map

$$
\Phi : E_1 \to E_2
$$

such that

$$
\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v) \in E_2,
$$

for any $v \in E_1$ and $g \in G$. Moreover we denote by $\text{Hom}_G(\rho_1, \rho_2)$ the set of homomorphism from $\rho_1$ to $\rho_2$. If a morphism $\Phi$ is bijective and its inverse, $\Phi^{-1}$, is also a morphism between $\rho_1$ and $\rho_2$ then $\Phi$ is called isomorphism.
Remark. It is easy to see that $\text{Hom}_G(\rho_1, \rho_2) \subset \text{Hom}(E_1, E_2)$ is a $k$-vector space.

**Exercise 1.** Let $G$ be and $\rho_1$, $\rho_2$ two $k$-representations of $G$, acting on the vector spaces $E_1$ and $E_2$.

i) Show that the action

$$
\rho_1 \oplus \rho_2 (g) : E_1 \oplus E_2 \to E_1 \oplus E_2
$$

$$
v \oplus w \mapsto \rho_1(g)v \oplus \rho_2(g)w
$$

is a $k$-representation of $G$. Moreover, if $\rho_1$, $\rho_2$ are finite dimensional, one has

$$
\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}.
$$

ii) Show that the action

$$
\rho_1 \otimes \rho_2 (g) : E_1 \otimes E_2 \to E_1 \otimes E_2
$$

$$
v \otimes w \mapsto \rho_1(g)v \otimes \rho_2(g)w
$$

is a $k$-representation of $G$. Moreover, if $\rho_1$, $\rho_2$ are finite dimensional, show that one has

$$
\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}.
$$

iii) Let $E_1'$ be the dual space of $E_1$. Show that the action

$$
\hat{\rho}_1(g) : E_1' \to E_1'
$$

$$
v \mapsto \lambda(v) \mapsto v \mapsto \lambda(\rho_1(g)^{-1})v
$$

is a $k$-representation of $G$. Moreover, if $\rho_1$ is finite dimensional, show that for any $g \in G$ one has

$$
\chi_{\hat{\rho}_1}(g) = \chi_{\rho_1}(g^{-1}).
$$

iv) Show that the action

$$
\rho(g) : \text{Hom}(E_1, E_2) \to \text{Hom}(E_1, E_2)
$$

$$
\Phi \mapsto \rho_2(g)\Phi \rho_1(g^{-1})
$$

is a $k$-representation of $G$ and $\rho \cong \hat{\rho}_1 \otimes \rho_2$ if $\rho_2$ is finite dimensional. Furthermore one has

$$
\text{Hom}(E_1, E_2)^G = \text{Hom}_G(\rho_1, \rho_2).
$$

Show that for any $g \in G$ one has

$$
\chi_\rho(g) = \chi_{\rho_1}(g^{-1}) \cdot \chi_{\rho_2}(g),
$$

if $\rho_1$, $\rho_2$ are finite dimensional.

**Exercise 2.** The Goal of this exercise is to prove Schur’s Lemma:

**Lemma 1** (Schur’s Lemma I,II). Let $G$ be a group, and let $k$ be an algebraically closed field.

i) If $\rho_1$ and $\rho_2$ are irreducible $k$-representations of $G$ which are not isomorphic, then

$$
\text{Hom}_G(\rho_1, \rho_2) = 0.
$$
ii) If \( \rho_1 \) and \( \rho_2 \) are finite-dimensional isomorphic irreducible \( k \)-representations of \( G \), then
\[
\dim(\text{Hom}_G(\rho_1, \rho_2)) = 1,
\]
and in fact if \( \rho \) is a finite-dimensional irreducible \( k \)-representation of \( G \) then
\[
\text{Hom}_G(\rho, \rho) = k \text{Id}_\rho,
\]

iii) Conversely, if \( \rho \) is a finite-dimensional semisimple \( k \)-representation of \( G \) such that \( \dim(\text{Hom}_G(\rho, \rho)) = 1 \), it follows that \( \rho \) is irreducible.

Proceed as follows

i) Let \( \pi, \rho \) two \( k \)-representation of \( G \) and assume that \( \pi \) irreducible. Show that if \( \Phi \in \text{Hom}_G(\pi, \rho) \), then either \( \Phi \) is injective or \( \Phi = 0 \). Similarly show that, if \( \Phi \in \text{Hom}_G(\rho, \pi) \), then either \( \Phi \) is surjective or \( \Phi = 0 \). Use this to prove the first part of Schur’s Lemma.

ii) For the second part we may assume without loss of generalities that \( \rho_1 = \rho_2 \). Show that, if \( \Phi \in \text{Hom}_G(\rho_1, \rho_1) \), then there exists \( \lambda \in k \) such that \( \Phi - \lambda \text{Id} \) is not injective. Use this to prove the second part of Schur’s Lemma.

iii) Let \( \rho \) be a finite-dimensional, semisimple \( k \)-representation of \( G \) such that \( \dim_G(\text{Hom}_G(\rho, \rho)) = 1 \). Let \( F \subset E \) be a subrepresentation and \( F_1 \) a complementary subrepresentation, so that \( E = F \oplus F_1 \). Show that
\[
\Phi : F \oplus F_1 \rightarrow F, \quad v \oplus w \mapsto v,
\]
is in \( \text{Hom}_G(\rho, \rho) \). Use this to prove part (iii) of Schur’s Lemma.

**Exercise 3.** Let \( V_m \) be the vector space of polynomials in \( \mathbb{C}[X, Y] \) which are homogeneous of degree \( m \). Consider the \( \mathbb{C} \)-representation
\[
\rho_m : \text{SL}_2(\mathbb{C}) \rightarrow \text{GL}(V_m)
\]
defined as follows: for any \( f \in V_m \) and any
\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}),
\]
we define
\[
(\rho_m(g)f)(X, Y) := f(aX + cY, bX + dY).
\]
The goal of this exercise is to show that \( \rho_m \) is an irreducible representation.

i) Show that \( \{X^iY^{m-i}\}_{i=0}^m \) is a basis for \( V_m \) and that \( \dim(V_m) = m + 1 \).

ii) Consider the subgroup \( T \subset \text{SL}_2(\mathbb{C}) \), defined as
\[
T := \{ t(\lambda) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^\times \}.
\]
Show that for any \( t(\lambda) \in T \) and any \( e_i := X^iY^{m-i} \), one has
\[
\rho_m(t(\lambda))e_i = \lambda^{2i-m} e_i.
\]
iii) Let \( 0 \subseteq F \subset V_m \) be a subspace of \( V_m \) stable under \( \rho_m \). Show that there exists \( 0 \leq i \leq m \) such that \( e_i \in F \).

iv) Show that, for any \( i = 0, ..., m \) one has

\[
\rho_m \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) e_i = \sum_{j=i}^{m-1} \binom{m-i}{j-i} e_j,
\]

and deduce that if \( e_i \in F \), then \( e_j \in F \) for any \( j \geq i \).

v) Using the action of the matrix

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},
\]

show that if \( e_i \in F \), then \( e_j \in F \) for any \( j \geq i \). Conclude the exercise.

Exercise 4. For this exercise assume the following fact

**Proposition 2** (Semisimplicity criterion.). Let \( G \) be a group, and let \( k \) be a field. A \( k \)-representation

\[
\rho : G \to \text{GL}(E),
\]

of \( G \) is semisimple if and only if, for any subrepresentation \( F_1 \subset E \) of \( \rho \), there exists a complementary subrepresentation, i.e. a \( G \)-stable subspace \( F_2 \subset E \) such that

\[
E = F_1 \oplus F_2.
\]

The goal of the exercise is to prove the following

**Theorem 3** (Maschke’s Theorem). Let \( G \) a finite group and let \( k \) be a field with characteristic not dividing \(|G|\). Then any \( k \)-representation

\[
\rho : G \to \text{GL}(E),
\]

is semisimple.

Proceed as follows:

i) Let \( F \subset E \) be a subrepresentation of \( \rho \). Let \( P_0 \in \text{Hom}(E, E) \) be a projection of \( E \) onto \( F \), i.e. \( P_0(E) = F \) and \( P_0|_F = \text{Id}_F \). Show that

\[
P := \frac{1}{|G|} \sum_{g \in G} g \cdot P_0 \in \text{Hom}_G(E, E),
\]

where \( g \cdot P_0(v) := \rho(g)P_0(\rho(g^{-1})v) \) for any \( g \in G \) and any \( v \in E \).

ii) Show that \( P \) is a projection of \( E \) onto \( F \) and that \( \text{Ker}(P) \) is a complementary subrepresentation of \( F \).

iii) Conclude.
Exercise 5. Let $G$ a finite group and let $k$ be a field with characteristic not dividing $|G|$. Prove that for any $k$-representation

$$\rho : G \to \text{GL}(E),$$

the map

$$P : \frac{1}{|G|} \sum_{g \in G} \rho(g) : E \to E$$

is a projection with image equal to $E^G$, the space of invariants of $G$. Moreover, if $E$ is finite dimensional, we have

$$\dim(E^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g).$$

Exercise 6. Let $G$ a finite group and let $k$ be an algebraically closed field with characteristic not dividing $|G|$. Use the previous exercise and Schur’s lemma to show that:

i) if $\rho_1$, $\rho_2$ are two finite-dimensional irreducible $k$-representations of $G$, one has

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1}) = \begin{cases} 0 & \text{if } \rho_1 \text{ is not isomorphic to } \rho_2, \\ 1 & \text{otherwise}, \end{cases}$$

ii) a finite-dimensional $k$-representation $\rho$ is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g)\chi_{\rho_1}(g^{-1}) = 1.$$

iii) Let $\pi, \rho$ two representations and assume $\pi$ irreducible, show that

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\chi_\pi(g^{-1}) = \dim(\text{Hom}_G(\pi, \rho)) =: n_\pi(\rho),$$

and deduce

$$\rho \cong \bigoplus_{\pi \in \text{irr}} n_\pi(\rho) \pi.$$

Exercise 7. Let $d \geq 2$, and consider the representation

$$\rho : S_d \to \text{GL}(\mathbb{C}^d),$$

where $S_d$ is the $d$-Symmetric Group.

i) Show that the subspace

$$H := \left\{ \sum_{i=1}^d a_ie_i : \sum_{i=1}^d a_i = 0 \right\}$$

is $S_d$-stable under the action of $S_d$. Conclude that $\rho$ is not irreducible.

ii) Show that the subrepresentation $\rho_H$ (see Definition 2 part (i)) is irreducible. Hint: First observe that for an unitary representation $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$. Then consider $\chi_\rho$ and compute

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g), \quad \frac{1}{|G|} \sum_{g \in G} |\chi_\rho(g)|^2.$$

Finally, prove that $\chi_{\rho_H}(g) = \chi_\rho(g) - 1$ and conclude.