D-MATH HS 2018 Prof. Emmanuel Kowalski
Exponential sums over Finite Fields. Exercise Sheet 4

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The aim of this exercise sheet is to recall the main definitions and properties of representations.

Definition 1. Let G be a group and k be a field. A *linear representations of* G, defined over k, is a group homomorphism

$$\rho : G \to \mathrm{GL}(E),$$

where E is a k-vector space. The dimension of E is called the *dimension of* ρ, or sometimes its *rank*, or *degree*. We will denote it $\dim(\rho)$. If ρ is a finite dimensional representation, we define *the character associated to* ρ,

$$\chi_\rho : \begin{array}{ccc} G & \to & k \\ g & \mapsto & \mathrm{Tr}(\rho(g)) \end{array}.$$

Definition 2. Let G be a group and $\rho : G \to \mathrm{GL}(E)$ a k-representation of G

i) We say that a subspace $F \subseteq E$, is said to be *stable under* ρ if $\rho(g) \cdot F \subseteq F$ for any $g \in G$. Then the k-representation

$$\rho_F : \begin{array}{ccc} G & \to & \mathrm{GL}(F) \\ g & \mapsto & \rho(g)_{|F} \end{array},$$

is called *subrepresentation of* ρ.

ii) We say that ρ is an *irreducible representation* if $E \neq 0$ and there is no stable subspace of E under ρ, except 0 and E itself.

iii) We say that ρ is *semisimple* if it can be written as a direct sum of subrepresentation, each of which is irreducible:

$$\rho \simeq \bigoplus_{i \in I} \rho_i.$$

Definition 3. Let G be a group and k a field. A *morphism*, or homomorphism, between two k-representation ρ_1 and ρ_2 of G, acting on the vector spaces E_1 and E_2, respectively, is a linear map

$$\Phi : E_1 \to E_2$$

such that

$$\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v) \in E_2,$$

for any $v \in E_1$ and $g \in G$. Moreover we denote by $\mathrm{Hom}_G(\rho_1, \rho_2)$ the set of homomorphism from ρ_1 to ρ_2.
If a morphism Φ is bijective and its inverse, Φ^{-1}, is also a morphism between ρ_1 and ρ_2 then Φ is called *isomorphism.*

Remark 1. It easy to see that $\mathrm{Hom}_G(\rho_1,\rho_2) \subset \mathrm{Hom}(E_1,E_2)$ is a k-vector space.

Exercise 1. Let G be and ρ_1, ρ_2 two k-representations of G, acting on the vector spaces E_1 and E_2.

i) Show that the action

$$\rho_1 \oplus \rho_2(g) : \begin{array}{ccc} E_1 \oplus E_2 & \to & E_1 \oplus E_2 \\ v \oplus w & \mapsto & \rho_1(g)v \oplus \rho_2(g)w \end{array}$$

is a k-representation of G. Moreover, if ρ_1, ρ_2 are finite dimensional, one has

$$\chi_{\rho_1\oplus\rho_2} = \chi_{\rho_1} + \chi_{\rho_2}.$$

ii) Show that the action

$$\rho_1 \otimes \rho_2(g) : \begin{array}{ccc} E_1 \otimes E_2 & \to & E_1 \otimes E_2 \\ v \otimes w & \mapsto & \rho_1(g)v \otimes \rho_2(g)w \end{array}$$

is a k-representation of G. Moreover, if ρ_1, ρ_2 are finite dimensional, show that one has

$$\chi_{\rho_1\otimes\rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}.$$

iii) Let E_1' be the dual space of E_1. Show that the action

$$\check{\rho}_1(g) : \begin{array}{ccc} E_1' & \to & E_1' \\ v \mapsto \lambda(v) & \mapsto & v \mapsto \lambda(\rho_1(g^{-1})v) \end{array}$$

is a k-representation of G. Moreover, if ρ_1 is finite dimensional, show that for any $g \in G$ one has

$$\chi_{\check{\rho}_1}(g) = \chi_{\rho_1}(g^{-1}).$$

iv) Show that the action

$$\rho(g) : \begin{array}{ccc} \mathrm{Hom}(E_1,E_2) & \to & \mathrm{Hom}(E_1,E_2) \\ \Phi & \mapsto & \rho_2(g)\Phi\rho_1(g^{-1}) \end{array}$$

is a k-representation of G and $\rho \cong \check{\rho}_1 \otimes \rho_2$ if ρ_2 is finite dimensional. Furthermore one has

$$\mathrm{Hom}(E_1,E_2)^G = \mathrm{Hom}_G(\rho_1,\rho_2).$$

Show that for any $g \in G$ one has

$$\chi_\rho(g) = \chi_{\rho_1}(g^{-1}) \cdot \chi_{\rho_2}(g),$$

if ρ_1, ρ_2 are finite dimensional.

Exercise 2. The Goal of this exercise is to prove Schur's Lemma:

Lemma 1 (Schur's Lemma I,II). *Let G be a group, and let k be an algebraically closed field.*

i) If ρ_1 and ρ_2 are irreducible k-representations of G which are not isomorphic, then

$$\mathrm{Hom}_G(\rho_1,\rho_2) = 0.$$

ii) If ρ_1 and ρ_2 are finite-dimensional isomorphic irreducible k-representations of G, then

$$\dim(\mathrm{Hom}_G(\rho_1, \rho_2)) = 1,$$

and in fact if ρ is a finite-dimensional irreducible k-representations of G then

$$\mathrm{Hom}_G(\rho, \rho)) = k\,\mathrm{Id}_\rho,$$

iii) Conversely, if ρ is a finite-dimensional semisimple k-representation of G such that $\dim(\mathrm{Hom}_G(\rho, \rho)) = 1$, it follows that ρ is irreducible.

Proceed as follows

i) Let π, ρ two k-representation of G and assume that π irreducible. Show that if $\Phi \in \mathrm{Hom}_G(\pi, \rho)$, then either Φ is injective or $\Phi = 0$. Similarly show that, if $\Phi \in \mathrm{Hom}_G(\rho, \pi)$, then either Φ is surjective or $\Phi = 0$. Use this two prove the first part of Schur's Lemma.

ii) For the second part we may assume without loss of generalities that $\rho_1 = \rho_2$. Show that, if $\Phi \in \mathrm{Hom}_G(\rho_1, \rho_1)$, then there exists $\lambda \in k$ such that $\Phi - \lambda\,\mathrm{Id}$ is not injective. Use this to prove the second part of Schur's Lemma.

iii) Let ρ be a finite-dimensional, semisimple k-representation of G such that $\dim_G(\mathrm{Hom}_G(\rho, \rho)) = 1$. Let $F \subset E$ be a subrepresentation and F_1 a complementary subrepresentation, so that $E = F \oplus F_1$. Show that

$$\Phi : \begin{array}{ccc} F \oplus F_1 & \to & F \\ v \oplus w & \mapsto & v \end{array},$$

is in $\mathrm{Hom}_G(\rho, \rho)$. Use this to prove part *(iii)* of Schur's Lemma

Exercise 3. Let V_m be the vector space of polynomials in $\mathbb{C}[X, Y]$ which are homogeneous of degree m. Consider the $\mathbb{C}$-representation

$$\rho_m : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V_m)$$

defined as follows: for any $f \in V_m$ and any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

we define

$$(\rho_m(g)f)(X, Y) := f(aX + cY, bX + dY).$$

The goal of this exercise is to show that ρ_m is an irreducible representation.

i) Show that $\{X^i Y^{m-i}\}_{i=0}^m$ is a basis for V_m and that $\dim(V_m) = m + 1$.

ii) Consider the subgroup $T \subset \mathrm{SL}_2(\mathbb{C})$, defined as

$$T := \left\{ t(\lambda) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^\times \right\}.$$

Show that for any $t(\lambda) \in T$ and any $e_i := X^i Y^{m-i}$, one has

$$\rho_m(t(\lambda))e_i = \lambda^{2i-m} e_i.$$

iii) Let $0 \subsetneq F \subset V_m$ be a subspace of V_m stable under ρ_m. Show that there exists $0 \leq i \leq m$ such that $e_i \in F$.

iv) Show that, for any $i = 0, ..., m$ one has

$$\rho_m\Big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\Big)e_i = \sum_{j=i}^{m-1} \binom{m-i}{j-i} e_j,$$

and deduce that if $e_i \in F$, then $e_j \in F$ for any $j \geq i$.

v) Using the action of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

show that if $e_i \in F$, then $e_j \in F$ for any $j \geq i$. Conclude the exercise.

Exercise 4. For this exercise assume the following fact

Proposition 2 (Semisimplicity criterion.)**.** *Let G be a group, and let k be a field. A k-representation*

$$\rho : G \to \mathrm{GL}(E),$$

of G is semisimple if and only if, for any subrepresentation $F_1 \subset E$ of ρ, there exists a complementary subrepresentation, i.e. a G-stable subspace $F_2 \subset E$ such that

$$E = F_1 \oplus F_2$$

The goal of the exercise is to prove the following

Theorem 3 (Maschke's Theorem)**.** *Let G a finite group and let k be a field with characteristic not dividing $|G|$. Then any k-representation*

$$\rho : G \to \mathrm{GL}(E),$$

is semisimple.

Proceed as follows:

i) Let $F \subset E$ be a subrepresentation of ρ. Let $P_0 \in \mathrm{Hom}(E, E)$ be a projection of E onto F, i.e. $P_0(E) = F$ and $P_{0|_F} = \mathrm{Id}_F$. Show that

$$P := \frac{1}{|G|} \sum_{g \in G} g \cdot P_0 \in \mathrm{Hom}_G(E, E),$$

where $g \cdot P_0(v) := \rho(g)P_0(\rho(g^{-1})v)$ for any $g \in G$ and any $v \in E$.

ii) Show that P is a projection of E onto F and that $\mathrm{Ker}(P)$ is a complementary subrepresentation of F.

iii) Conclude.

Exercise 5. Let G a finite group and let k be a field with characteristic not dividing $|G|$. Prove that for any k-representation

$$\rho : G \to \mathrm{GL}(E),$$

the map

$$P : \frac{1}{|G|} \sum_{g \in G} \rho(g) : E \to E$$

is a projection with image equal to E^G, the space of invariants of G. Moreover, if E is finite dimensional, we have

$$\dim(E^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g).$$

Exercise 6. Let G a finite group and let k be an algebraically closed field with characteristic not dividing $|G|$. Use the previous exercise and Schur's lemma to show that:

i) if ρ_1, ρ_2 are two finite-dimensional irreducible k-representations of G, one has

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1}) = \begin{cases} 0 & \text{if } \rho_1 \text{ is not isomorphic to } \rho_2, \\ 1 & \text{otherwise,} \end{cases}$$

ii) a finite-dimensional k-representation ρ is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g)\chi_{\rho_1}(g^{-1}) = 1.$$

iii) Let π, ρ two representations and assume π irreducible, show that

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\chi_\pi(g^{-1}) = \dim(\mathrm{Hom}_G(\pi, \rho)) =: n_\pi(\rho),$$

and deduce

$$\rho \cong \bigoplus_{\pi \text{ irr}} \pi^{\oplus n_\pi(\rho)}.$$

Exercise 7. Let $d \geq 2$, and consider the representation

$$\rho : \begin{array}{ccc} S_d & \to & \mathrm{GL}(\mathbb{C}^d) \\ \sigma & \mapsto & e_i \mapsto e_{\sigma(i)} \end{array},$$

where S_d is the d-Symmetric Group.

i) Show that the subspace

$$H := \Big\{ \sum_{i=1}^{d} a_i e_i : \sum_{i=1}^{d} a_i = 0 \Big\}$$

is S_d-stable under the action of S_d. Conclude that ρ is not irreducible.

ii) Show that the subrepresentation ρ_H (see Definition 2 part (i)) is irreducible. **Hint:** First observe that for an unitary representation $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$. Then consider χ_ρ and compute

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g), \quad \frac{1}{|G|} \sum_{g \in G} |\chi_\rho(g)|^2.$$

Finally, prove that $\chi_{\rho_H}(g) = \chi_\rho(g) - 1$ and conclude.