D-MATH HS 2018 PROF. EMMANUEL KOWALSKI EXPONENTIAL SUMS OVER FINITE FIELDS. EXERCISE SHEET 6

We write χ for multiplicative characters and ψ for additive characters; the additive characters are always assumed to be non-trivial, but not necessarily the multiplicative characters.

In the exercises, it is allowed to use the version of the Riemann Hypothesis (Theorem 1), of the trace formula, etc, for finite extensions of $\mathbb{F}_p$.

One may also use the following fact: for any ℓ-adic sheaf ρ modulo p and any invertible matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ modulo p, there exists an ℓ-adic sheaf $\gamma^*\rho$, with the same conductor, such that

$$t_{\gamma^*\rho}(x) = t_\rho\Big(\frac{ax+b}{cx+d}\Big)$$

for all $x \in \mathbb{F}_p$. (If $cx + d = 0$, then the right-hand side is the value of the trace function at infinity.)

Exercise 1. If ρ_1 is an ℓ-adic sheaf modulo p that is ramified at x and ρ_2 is an ℓ-adic sheaf modulo p that is unramified at x, then $\rho_1 \otimes \rho_2$ is ramified at x and

$$t_{\rho_1\otimes\rho_2}(x) = t_{\rho_1}(x)t_{\rho_2}(x).$$

Exercise 2. For the following exponential sums modulo p, compute the dimension of H^1_c and deduce an upper bound on the sums:

$$\sum_{x\in\mathbb{F}_p} \psi(ax + x^3), \qquad a \in \mathbb{F}_p^\times$$

$$\sum_{x\in\mathbb{F}_p} \chi(x^3 + ax + b), \qquad (a, b) \in \mathbb{F}_p, \quad \chi \neq 1$$

$$\sum_{x\in\mathbb{F}_p^\times} \chi(x)\psi(ax + x^{-1}), \qquad a \in \mathbb{F}_p^\times, \quad \chi \neq 1$$

$$\sum_{x\in\mathbb{F}_p\setminus\{0,1,2\}} \psi(1/x + 1/(x-1) + 1/(x-2))$$

$$\sum_{x\in\mathbb{F}_p} \chi_1(x + 1/x)\psi(x^2 + 1).$$

Exercise 3. Let ρ be a geometrically irreducible ℓ-adic sheaf modulo p.

(1) If ρ is unramified everywhere, then ρ is geometrically trivial.
(2) If ρ is unramified everywhere except at a point x, and if ρ is tamely ramified at x, then ρ is geometrically trivial.

(Apply the Euler-Poincaré formula.)

Exercise 4. Let C be a fixed constant. The goal of this exercise is to prove that there are only finitely many geometrically irreducible ℓ-adic sheaves modulo p, up to geometric isomorphism, with conductor $\leq C$.

(1) Prove that if E is a finite-dimensional real euclidean space and $\xi > 0$ is a real number, and if Y is a subset of vectors of length 1 in E such that $\langle x, y\rangle \leq 1 - \xi$ for all $x \neq y$ in Y, then the set Y is finite. (Consider small "caps" around each point $x \in Y$.)
(2) Deduce the statement for p large enough in terms of C.

(3) By passing to a finite extension of $\mathbb{F}_p$, explain how to deduce the statement for any p.

Exercise 5. Using Katz's criterion for irreducibility, prove that the Kloosterman sheaf $\mathcal{K}\ell_2$ modulo p with trace function over $\mathbb{F}_{p^n}$ given by

$$-\frac{1}{p^{n/2}}\sum_{x\in\mathbb{F}_{p^n}^\times}\psi(\mathrm{Tr}(ax+1/x))$$

is geometrically irreducible.

Exercise 6. For all $a\in\mathbb{F}_p$, let

$$\mathrm{Kl}_3(a;p)=\frac{1}{p}\sum_{xyz=a}e\Bigl(\frac{x+y+z}{p}\Bigr)$$

(where (x,y,z) are in $\mathbb{F}_p$).

(1) Prove that for a, b, c in $\mathbb{F}_p$ with $ab\neq 0$, we have

$$\sum_{x\in\mathbb{F}_p}\mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)}e\Bigl(\frac{cx}{p}\Bigr)=\sum_{\substack{t\in\mathbb{F}_p\\ t\neq 0,-c}}\mathrm{Kl}_2\Bigl(\frac{a}{t};p\Bigr)\mathrm{Kl}_2\Bigl(\frac{b}{t+c};p\Bigr).$$

(2) Using the existence of the Kloosterman sheaf $\mathcal{K}\ell_2$ modulo p with trace function $-\mathrm{Kl}_2(a;p)$, show that there exists a constant C such that

$$\Bigl|\sum_{x\in\mathbb{F}_p}\mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)}e\Bigl(\frac{cx}{p}\Bigr)\Bigr|\leq C\sqrt{p}$$

for all p and all (a,b,c), unless $a=b$ and $c=0$.

Exercise 7.

(1) Let ρ_1 and ρ_2 two geometrically irreducible ℓ-adic sheaves modulo p, both of which are ramified at most at 0 and ∞, and both of which are tamely ramified. Prove that if ρ_1 and ρ_2 are not geometrically isomorphic, then

$$\sum_{x\in\mathbb{F}_p}t_{\rho_1}(x)\overline{t_{\rho_2}(x)}=0,$$

(2) Let ρ be a geometrically irreducible ℓ-adic sheaf modulo p ramified at most at 0 and ∞ and tamely ramified. Show that there exists a multiplicative character χ such that ρ is geometrically isomorphic to $\mathcal{L}_{\chi(X)}$. (Otherwise, show using (1) and characters of $\mathbb{F}_p^\times$ that the trace function would be 0 over any finite extension of $\mathbb{F}_p$.)

(3) Show that the conclusion of (2) does not hold if one removes the assumption that ρ is tamely ramified.

Exercise 8. Arguing along the same lines as the previous exercise, show that if ρ is a geometrically irreducible ℓ-adic sheaf modulo p that is ramified at most at ∞, and has Swan conductor ≤ 1 at infinity, then ρ is geometrically isomorphic to a sheaf $\mathcal{L}_{\psi(X)}$ for some additive character ψ.