D-MATH HS 2018 Prof. Emmanuel Kowalski Exponential sums over Finite Fields. Exercise Sheet 6

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Exercise 1 Let ρ_1, ρ_2 be two ℓ-adic representation modulo p. By definition of a trace function attached to an ℓ-adic representation modulo p, to solve the exercise it is enough to check that

$$(V_1 \otimes V_2)^{I_x} = V_1^{I_x} \otimes V_2^{I_x}$$

when x is a singular point for ρ_1 and an unramified point for ρ_2. Let $v_1 \in V_1^{I_x}$ and let $v_2 \in V_2^{I_x}$, then for any $g \in I_x$ we have

$$\rho_1 \otimes \rho_2(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) = v_1 \otimes v_2.$$

Hence $(V_1 \otimes V_2)^{I_x} \supseteq V_1^{I_x} \otimes V_2^{I_x}$. Let $w \in (V_1 \otimes V_2)^{I_x}$, then we can write w as

$$\sum_{j=1}^{n} v_{1,j} \otimes e_{2,j}$$

where $v_{1,j} \in V_1$ for any j and $\{e_{2,1}, ..., e_{2,n}\}$ is a basis for V_2. Then for any $g \in I_x$,one has

$$\rho_1 \otimes \rho_2(g)(w) = \sum_{j=1}^{n} \rho_1(g)(v_{1,j}) \otimes \rho_2(g)(e_{2,j}) = \sum_{j=1}^{n} \rho_1(g)(v_{1,j}) \otimes e_{2,j}$$

where the last step follows from the fact that $V_2^{I_x} = V_2$ since x is an unramified point for ρ_2. Then

$$\sum_{j=1}^{n} (v_{1,j} - \rho_1(g)(v_{1,j})) \otimes e_{2,j} = 0.$$

Thus

$$v_{1,j} - \rho_1(g)(v_{1,j}) = 0,$$

for any j and this implies that $v_{1,j} \in V_1^{I_x}$ for any j. Hence $w \in V_1^{I_x} \otimes V_2^{I_x}$ as we wanted.

Exercise 2. We discuss just

$$\sum_{x \in \mathbb{F}_q} \psi(x + ax^3) \quad a \in \mathbb{F}_q^\times, \quad \sum_{x \in \mathbb{F}_q} \chi(x)\psi(x + x^{-1}), a \in \mathbb{F}_q^\times, \quad \chi \neq 1,$$

the others are analogues.

i) We know that the function $\psi(x+ax^3)$ is the trace function attached to the Artin-Schreier sheaf $\mathcal{L}_{\psi(T+aT^3)}$. We know from the lectures that $\mathcal{L}_{\psi(T+aT^3)}$ is irreducible and not geometrically trivial. Moreover $\mathrm{rank}(\mathcal{L}_{\psi(T+aT^3)}) = 1$, $\mathrm{sing}(\mathcal{L}_{\psi(T+aT^3)}) = \{\infty\}$ and if $a \neq 0$ then $\mathrm{Swan}_\infty(\mathcal{L}_{\psi(T+aT^3)}) = 3$. An application of the Euler-Poincaré formula leads to

$$-h^1 = \mathrm{rank}(\mathcal{L}_{\psi(T+aT^3)}) - \mathrm{Swan}_\infty(\mathcal{L}_{\psi(T+aT^3)}) = -2.$$

Thus the Riemann Hypothesis over finite fields implies that

$$\Big|\sum_{x\in\mathbb{F}_q}\psi(x+ax^3)\Big| \leq 2\sqrt{p}$$

for any $a \in \mathbb{F}_q^\times$.

ii) First recall that:

a) The function $\psi(x+ax^{-1})$ is the trace function attached to the Artin-Schreier sheaf $\mathcal{L}_{\psi(T+aT^{-1})}$. We know from the lectures that $\mathcal{L}_{\psi(T+aT^{-1})}$ is irreducible and not geometrically trivial. Moreover $\mathrm{rank}(\mathcal{L}_{\psi(T+aT^{-1})}) = 1$ and if $a \neq 0$ then $\mathrm{sing}(\mathcal{L}_{\psi(T+aT^{-1})}) = \{0,\infty\}$, $\mathrm{Swan}_\infty(\mathcal{L}_{\psi(T+aT^{-1})}) = 1$, $\mathrm{Swan}_0(\mathcal{L}_{\psi(T+aT^{-1})}) = 1$ and $\mathrm{drop}_0(\mathcal{L}_{\psi(T+aT^{-1})}) = 1$.

b) The function $\chi(x)$ is the trace function attached to the Kummer sheaf $\mathcal{L}_{\chi(T)}$. We know from the lectures that if $\chi \neq 1$ then $\mathcal{L}_{\chi(T)}$ is irreducible and not geometrically trivial. Moreover $\mathrm{rank}(\mathcal{L}_{\chi(T)}) = 1$, $\mathrm{sing}(\mathcal{L}_{\chi(T)}) = \{0,\infty\}$, $\mathrm{Swan}_\infty(\mathcal{L}_{\chi(T)}) = 0$, $\mathrm{Swan}_0(\mathcal{L}_{\chi(T)}) = 0$ and $\mathrm{drop}_0(\mathcal{L}_{\chi(T)}) = 1$.

c) If ρ_1, ρ_2 are two ℓ-adic representation modulo p then $\mathrm{rank}(\rho_1\otimes\rho_2) = \mathrm{rank}(\rho_1)\,\mathrm{rank}(\rho_2)$ and $\mathrm{sing}(\rho_1\otimes\rho_2) \subset \mathrm{sing}(\rho_1)\cup\mathrm{sing}(\rho_2)$. Moreover for any singular point x we have that $\mathrm{Swan}_x(\rho_1\otimes\rho_2) \leq \mathrm{rank}(\rho_1)\,\mathrm{rank}(\rho_2)(\mathrm{Swan}_x(\rho_1)+\mathrm{Swan}_x(\rho_2))$.

Let us consider $\mathcal{G} := \mathcal{L}_{\psi(T+aT^{-1})} \otimes \mathcal{L}_{\chi(T)}$. Than $\mathrm{rank}(\mathcal{G}) = 1$, moreover $\mathcal{G}$ is not geometrically trivial since $\mathcal{L}_{\psi(T+aT^{-1})} \not\cong_{\mathrm{geom}} \mathcal{L}_{\chi(T)}$ (indeed, $\mathcal{L}_{\psi(T+aT^{-1})}$ is wild ramified in $0,\infty$ while $\mathcal{L}_{\chi(T)}$ is tame everywhere). On the other hand combining (*a*), (*b*) and (*c*) we get

$$\mathrm{sing}(\mathcal{G}) \subset \{0,\infty\}, \quad \mathrm{Swan}_0(\mathcal{G}), \mathrm{Swan}_\infty(\mathcal{G}) \leq 1, \quad \mathrm{drop}_0(\mathcal{G}) = 1.$$

Applying the Euler-Poincaré formula one gets

$$-h^1 = \mathrm{rank}(\mathcal{G}) - \mathrm{drop}_0(\mathcal{G}) - \mathrm{Swan}_0(\mathcal{G}) - \mathrm{Swan}_\infty(\mathcal{G}) = -2.$$

Thus the Riemann Hypothesis over finite fields implies that

$$\Big|\sum_{x\in\mathbb{F}_q}\chi(x)\psi(x+ax^{-1})\Big| \leq 2\sqrt{p}$$

for any $a \in \mathbb{F}_q^\times$ and $\chi \neq 1$.

Exercise 3.

i) The Euler-Poincaré formula tells us that

$$h^0 - h^1 + h^2 = \mathrm{rank}(\rho) - \sum_{x\in\overline{\mathbb{F}}_q}\mathrm{drop}_x(\rho) - \sum_{x\in\overline{\mathbb{F}}_q\cup\{\infty\}}\mathrm{Swan}_x(\rho).$$

If ρ is unramified everywhere then for any $x \in \overline{\mathbb{F}}_q \cup \{\infty\}$ one has

$$\mathrm{drop}_x(\rho) = \mathrm{Swan}_x(\rho) = 0.$$

Moreover we know from the lecture that for a geometrically irreducible ℓ-adic representation modulo p one has that

$$h^0 = 0, \quad h^2 = \begin{cases} 1, & \text{if } \rho \text{ is geometrically trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, one gets

$$-h^1 + h^2 = \mathrm{rank}(\rho) > 0.$$

This implies that $h^1 = 0$, $h^2 = 1$, i.e. ρ is geometrically irreducible.

ii) First of all, we claim that without loss of generality we may assume that $x = \infty$: we can replace ρ with $\gamma^*\rho$ where γ is the Möbius transformation

$$\gamma : y \mapsto \frac{y}{yx^{-1} - 1}.$$

Indeed, one has that $\mathrm{sing}(\gamma^*\rho) = \gamma^{-1}(\mathrm{sing}(\rho)) = \{\gamma^{-1}(x)\} = \{\infty\}$. Moreover ρ is geometrically trivial if and only if $\gamma^*\rho$ is geometrically trivial and this proves the claim. Now, let ρ be an ℓ-adic representation modulo p unramified on $\mathbb{A}_{\overline{\mathbb{F}}_q}$ and tame at ∞. One starts again with the Euler-Poincaré formula getting

$$h^0 - h^1 + h^2 = \mathrm{rank}(\rho) - \sum_{x \in \overline{\mathbb{F}}_q} \mathrm{drop}_x(\rho) - \sum_{x \in \overline{\mathbb{F}}_q \cup \{\infty\}} \mathrm{Swan}_x(\rho).$$

For any $x \in \mathbb{A}_{\overline{\mathbb{F}}_q}$ one has

$$\mathrm{drop}_x(\rho) = \mathrm{Swan}_x(\rho) = 0,$$

Moreover $\mathrm{Swan}_\infty(\rho) = 0$, since ρ is tame at ∞. Hence, one obtains

$$-h^1 + h^2 = \mathrm{rank}(\rho) > 0.$$

Using the same argument as in part (i) we conclude the exercise.

Exercise 4.

i) One has that for any $x \in Y$

$$Y \cap B_{\xi/2}(x) = \{x\},$$

where $B_{\xi/2}(x) := \{y \in E : \|x - y\| \leq \xi/2\}$. Indeed if $y \in B_{\xi/2}(x)$ and $\|y\| = 1$, then

$$\xi/2 \geq \|x - y\| = \|x\| + \|y\| - 2\langle x, y\rangle = 2 - 2\langle x, y\rangle,$$

which means

$$\langle x, y\rangle \geq 1 - \xi/4,$$

thus $y \notin Y$. Now observe that

$$|Y| = \sum_{x \in Y} 1 = \sum_{x \in Y} \frac{\mu(B_{\xi/2}(x) \cap \mathbb{S}^n)}{\mu(B_{\xi/2}(x) \cap \mathbb{S}^n)},$$

where $\mathbb{S}^n = \{e \in E : \|e\| = 1\}$ and μ is the $n-1$ dimensional Lebesgue measure. On the other hand $\mu(B_{\xi/2}(x) \cap \mathbb{S}^n) =: \gamma > 0$ is independent on x and

$$\sum_{x \in Y} \mu(B_{\xi/2}(x) \cap \mathbb{S}^n) = \mu\Big(\bigcup_{x \in Y} B_{\xi/2}(x) \cap \mathbb{S}^n\Big) \leq \mu(\mathbb{S}^n) < \infty,$$

since $B_{\xi/2}(x) \cap B_{\xi/2}(y) = \emptyset$ for $x, y \in Y$ with $x \neq y$. Then we conclude that

$$|Y| \leq \frac{\mu(\mathbb{S}^n)}{\gamma} < \infty$$

as we wanted.

ii) Let us consider the space $E = \mathbb{C}^p$. Consider the set

$$\mathcal{T}_C := \{\rho : \rho \text{ is a geometrically irreducible } \ell\text{-adic representation with } c(\rho) \leq C\},$$

with the equivalence relation $\rho_1 \sim \rho_2$ if and only if $\rho_1 \cong_{\mathrm{geom}} \rho_2$. For any equivalence class $[\rho]$ of trace functions we associate a vector in E in the following way: pick $\rho \in [\rho]$, and define

$$v_{[\rho]} := \frac{1}{\sum_{x \in \mathbb{F}_p} |t_\rho(x)|^2} (t_\rho(x))_{x \in \mathbb{F}_p}.$$

Notice that $\|v_{[\rho]}\| = 1$ independently of the choice of ρ in $[\rho]$. Let $[\rho_1], [\rho_2]$ be two different classes, then

$$\langle v_{[\rho_1]}, v_{[\rho_2]} \rangle = \frac{\sum_{x \in \mathbb{F}_p} t_{\rho_1}(x)\overline{t_{\rho_2}(x)}}{\sum_{x \in \mathbb{F}_p} |t_\rho(x)|^2 \times \sum_{x \in \mathbb{F}_p} |t_\rho(x)|^2} \leq \frac{10C^4}{p^2}$$

thanks to the Riemann Hypothesis over finite fields and the Katz's criterion for irreducibility. Thus for p large enough

$$\frac{10C^4}{p^2} < 1$$

and we can apply part (i) to the set $\mathcal{T}_C/\sim$.

iii) For any $[\rho] \in \mathcal{T}_C/\sim$ and any $n \geq 1$ we define

$$v_{[\rho;n]} := \frac{1}{\sum_{x \in \mathbb{F}_{p^n}} |t_\rho(x;n)|^2} (t_\rho(x;n))_{x \in \mathbb{F}_{p^n}}.$$

Then for any $[\rho_1], [\rho_2]$ two different classes, one gets

$$\langle v_{[\rho_1;n]}, v_{[\rho_2;n]} \rangle = \frac{\sum_{x \in \mathbb{F}_{p^n}} t_{\rho_1}(x;n)\overline{t_{\rho_2}(x;n)}}{\sum_{x \in \mathbb{F}_{p^n}} |t_\rho(x;n)|^2 \times \sum_{x \in \mathbb{F}_{p^n}} |t_{\rho;n}(x)|^2} \leq \frac{10C^4}{p^{2n}}.$$

Thus for n large enough

$$\frac{10C^4}{p^{2n}} < 1$$

and we can apply part (i) to the set $\mathcal{T}_C/\sim$.

Exercise 5. One has that

$$\frac{1}{p^n}\sum_{a\in\mathbb{F}_p}\frac{1}{p^n}\Big|\sum_{x\in\mathbb{F}_p^\times}\psi(\operatorname{Tr}(ax+x^{-1}))\Big|^2 = \frac{1}{p^{2n}}\sum_{a\in\mathbb{F}_p}\sum_{x,y\in\mathbb{F}_p^\times}\psi(\operatorname{Tr}(ax+x^{-1}))\psi(-\operatorname{Tr}(ay+y^{-1}))$$
$$= \frac{1}{p^n}\sum_{x,y\in\mathbb{F}_p^\times}\psi(\operatorname{Tr}(x^{-1}-y^{-1}))\frac{1}{p^n}\sum_{a\in\mathbb{F}_p}\psi(\operatorname{Tr}(a(x-y)))$$
$$= 1-p^{-n}$$

where the last step follows from the orthogonality relation of the additive characters. Then the Kloosterman sheaf, $\mathcal{K}\ell_2$, is irreducible thank to Katz's criterion for irreducibility.

Exercise 6.

i) One has that

$$\sum_{x\in\mathbb{F}_p}\mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)}e\Big(\frac{cx}{p}\Big) = \frac{1}{p^2}\sum_{x\in\mathbb{F}_p}\sum_{r,s\in\mathbb{F}_p^\times}\sum_{f,g\in\mathbb{F}_p^\times}e\Big(\frac{r+s-f-g+x(c+a\overline{rs}-b\overline{fg})}{p}\Big)$$
$$= \frac{1}{p}\sum_{r,s\in\mathbb{F}_p^\times}\sum_{f,g\in\mathbb{F}_p^\times}e\Big(\frac{r+s-f-g}{p}\Big)\frac{1}{p}\sum_{x\in\mathbb{F}_p}e\Big(\frac{x(c+a\overline{rs}-b\overline{fg})}{p}\Big)$$
$$= \frac{1}{p}\sum_{\substack{r,s\in\mathbb{F}_p^\times\\ a\overline{rs}\neq -c}}\sum_{f,g\in\mathbb{F}_p^\times}e\Big(\frac{r+s-f-g}{p}\Big)\frac{1}{p}\sum_{x\in\mathbb{F}_p}e\Big(\frac{x(c+a\overline{rs}-b\overline{fg})}{p}\Big),$$

where the last step follows from the fact that if $a\overline{rs}=-c$, then

$$\sum_{x\in\mathbb{F}_p}e\Big(\frac{-xb\overline{fg}}{p}\Big)=0,$$

since $b\overline{fg}\neq 0$. Now we change the variable $t=a\overline{rs}$ thus we obtain

$$\sum_{x\in\mathbb{F}_p}\mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)}e\Big(\frac{cx}{p}\Big) = \frac{1}{p}\sum_{t\in\mathbb{F}_p\setminus\{0,-c\}}\sum_{r\in\mathbb{F}_p^\times}\sum_{f,g\in\mathbb{F}_p^\times}e\Big(\frac{r+a\overline{tr}-f-g}{p}\Big)\frac{1}{p}\sum_{x\in\mathbb{F}_p}e\Big(\frac{x(c+t-b\overline{fg})}{p}\Big)$$
$$= \frac{1}{p}\sum_{t\in\mathbb{F}_p\setminus\{0,-c\}}\sum_{r\in\mathbb{F}_p^\times}e\Big(\frac{r+a\overline{tr}}{p}\Big)\sum_{\substack{f,g\in\mathbb{F}_p^\times\\ fg=b/(c+t)}}e\Big(\frac{-f-g}{p}\Big)$$
$$= \sum_{t\in\mathbb{F}_p\setminus\{0,-c\}}\Big(\frac{1}{\sqrt{p}}\sum_{r\in\mathbb{F}_p^\times}e\Big(\frac{r+a\overline{tr}}{p}\Big)\Big)\times\Big(\frac{1}{\sqrt{p}}\sum_{\substack{f,g\in\mathbb{F}_p^\times\\ fg=b\overline{c+t}}}e\Big(\frac{f+\overline{f}b\overline{(c+t)}}{p}\Big)$$
$$= \sum_{t\in\mathbb{F}_p\setminus\{0,-c\}}\mathrm{Kl}_2\Big(\frac{a}{t};p\Big)\mathrm{Kl}_2\Big(\frac{b}{t+c};p\Big),$$

as we wanted.

i) Since

$$\sum_{x\in\mathbb{F}_p}\mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)}e\Big(\frac{cx}{p}\Big) = \sum_{t\in\mathbb{F}_p\setminus\{0,-c\}}\mathrm{Kl}_2\Big(\frac{a}{t};p\Big)\mathrm{Kl}_2\Big(\frac{b}{t+c};p\Big),$$

it is enough to check that $\gamma_a^*\mathcal{K}\ell_2 \not\cong_{\text{geom}} \gamma_{b,c}^*\mathcal{K}\ell_2$ unless $a=b$ and $c=0$, where γ_a and $\gamma_{b,c}$ are the Möebius transformations

$$\gamma_a : t \mapsto \frac{a}{t}, \quad \gamma_{b,c} : t \mapsto \frac{b}{t+c}.$$

Assume first $c=0$ and $a \neq b$, then

$$\begin{aligned}\sum_{t\in\mathbb{F}_p^\times} \mathrm{Kl}_2\left(\frac{a}{t};p\right)\mathrm{Kl}_2\left(\frac{b}{t};p\right) &= \frac{1}{p}\sum_{t\in\mathbb{F}_p^\times}\sum_{x\in\mathbb{F}_p^\times}\sum_{y\in\mathbb{F}_p^\times} e\Big(\frac{x+a\overline{tx}-y-b\overline{ty}}{p}\Big)\\ &= \sum_{x\in\mathbb{F}_p^\times}\sum_{y\in\mathbb{F}_p^\times} e\Big(\frac{x-y}{p}\Big)\frac{1}{p}\sum_{t\in\mathbb{F}_p^\times} e\Big(\frac{\overline{t}(a\overline{x}-b\overline{y})}{p}\Big)\end{aligned}$$

Using the orthogonal relation of the additive character we get

$$\begin{aligned}\sum_{t\in\mathbb{F}_p^\times} \mathrm{Kl}_2\left(\frac{a}{t};p\right)\mathrm{Kl}_2\left(\frac{b}{t};p\right) &= -\frac{1}{p}\sum_{x\in\mathbb{F}_p^\times}\sum_{\substack{y\in\mathbb{F}_p^\times\\ y\neq xb\overline{a}}} e\Big(\frac{x-y}{p}\Big) + \sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x(1+b\overline{a})}{p}\Big)\\ &= -\frac{1}{p}\sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x}{p}\Big)\sum_{\substack{y\in\mathbb{F}_p^\times\\ y\neq xb\overline{a}}} e\Big(-\frac{y}{p}\Big) + \sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x(1+b\overline{a})}{p}\Big)\end{aligned}$$

Now we have that

$$\sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x(1+b\overline{a})}{p}\Big) = -1$$

since we are assuming $a \neq b$. Moreover one has that

$$-\frac{1}{p}\sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x}{p}\Big)\sum_{\substack{y\in\mathbb{F}_p^\times\\ y\neq xb\overline{a}}} e\Big(-\frac{y}{p}\Big) = \frac{1}{p}\sum_{x\in\mathbb{F}_p^\times} e\Big(\frac{x}{p}\Big)\Big(1+e\Big(\frac{xb\overline{a}}{p}\Big)\Big) = -\frac{2}{p}$$

thanks to the fact that $a \neq b$. Thus if $a \neq b$ and $c=0$ we get

$$\sum_{x\in\mathbb{F}_p} \mathrm{Kl}_3(ax;p)\overline{\mathrm{Kl}_3(bx;p)} = -1-\frac{2}{p}.$$

Assume now $c \neq 0$, then $\mathrm{sing}(\gamma_a^*\mathcal{K}\ell_2) = \{0,\infty\}$ while $\mathrm{sing}(\gamma_{b,c}^*\mathcal{K}\ell_2) = \{-c,\infty\}$. Then $\gamma_a^*\mathcal{K}\ell_2 \not\cong_{\text{geom}} \gamma_{b,c}^*\mathcal{K}\ell_2$ since $\mathrm{sing}(\gamma_a^*\mathcal{K}\ell_2) \neq \mathrm{sing}(\gamma_{b,c}^*\mathcal{K}\ell_2)$ if $c \neq 0$. Applying the Riemann Hypothesis over finite fields one gets

$$\sum_{t\in\mathbb{F}_p\setminus\{0,-c\}} \mathrm{Kl}_2\left(\frac{a}{t};p\right)\mathrm{Kl}_2\left(\frac{b}{t+c};p\right) \leq 5c(\gamma_a^*\mathcal{K}\ell_2)^2 c(\gamma_{b,c}^*\mathcal{K}\ell_2)^2\sqrt{p}.$$

Since $c(\gamma_a^*\mathcal{K}\ell_2) = c(\gamma_{b,c}^*\mathcal{K}\ell_2) = c(\mathcal{K}\ell_2)$, one gets

$$\sum_{t\in\mathbb{F}_p\setminus\{0,-c\}} \mathrm{Kl}_2\left(\frac{a}{t};p\right)\mathrm{Kl}_2\left(\frac{b}{t+c};p\right) \leq 5c(\mathcal{K}\ell_2)^4\sqrt{p},$$

as we wanted.

Exercise 7.

i) Let ρ_1 and ρ_2 be two geometrically irreducible ℓ-adic sheaf modulo p, both of which are ramified at most at 0 and ∞, and both of which are tamely ramified everywhere. Moreover assume that $\rho_1 \not\cong_{\text{geom}} \rho_2$. Applying the the Euler-Poincaré formula one gets

$$h^0 - h^1 + h^2 = \text{rank}(\rho_1 \otimes \rho_2) - \text{drop}_0(\rho_1 \otimes \rho_2).$$

We know from the lecture that $h^0 = 0$ and that $h^2 = 0$ since ρ_1 and ρ_2 are two geometrically irreducible ℓ-adic sheaf modulo p and $\rho_1 \not\cong_{\text{geom}} \rho_2$. Thus

$$-h^1 = \text{rank}(\rho_1 \otimes \rho_2) - \text{drop}_0(\rho_1 \otimes \rho_2).$$

On the other hand $\text{drop}_0(\rho_1 \otimes \rho_2) \leq \text{rank}(\rho_1 \otimes \rho_2)$ by definition of the drop. Hence,

$$-h^1 \geq 0;$$

which implies $h^1 = 0$ and this conclude the proof of (i).

ii) Let ρ be a geometrically irreducible ℓ-adic sheaf modulo p, which is ramified at most at 0 and ∞, and tamely ramified everywhere. By contradiction, assume that $\rho \not\cong \mathcal{L}_{\chi(T)}$ for any multiplicative character χ. Then for any $n \geq 1$ and any $x \in \mathbb{F}_p^n$ one would have that

$$\begin{aligned}
t_\rho(x;n) &= \sum_{y \in \mathbb{F}_p^n} t_\rho(y;n) \frac{1}{p-1} \sum_\chi \chi(xy^{-1}) \\
&= \sum_{y \in \mathbb{F}_p^n} t_\rho(y;n) \frac{1}{p-1} \sum_\chi \chi(x)\overline{\chi(y)} \\
&= \frac{1}{p-1} \sum_{y \in \mathbb{F}_p^n} \sum_\chi t_\rho(y;n)\chi(x)\overline{\chi(y)} \\
&= \frac{1}{p-1} \sum_\chi \chi(x) \sum_{y \in \mathbb{F}_p^n} t_\rho(y;n)\overline{\chi(y)} \\
&= 0,
\end{aligned}$$

where in the last step we used part (i) since $\rho \not\cong \mathcal{L}_{\chi(T)}$ for any multiplicative character χ and $\mathcal{L}_{\chi(T)}$ is ramified only at 0 and ∞ and it is tame everywhere. Now this would implies that

$$\sum_{x \in \mathbb{F}_{p^n}} |t_\rho(x;n)|^2 = 0$$

for any $n \geq 1$ and this is not possible since the Riemann Hypothesis over finite fields tells us that

$$\Big| \sum_{x \in \mathbb{F}_{p^n}} |t_\rho(x;n)|^2 - p \Big| \leq c(\rho) p^{n/2},$$

for any geometrically irreducible ℓ-adic representation modulo p.

iii) It is enough to take the Artin-Schreier sheaf $\mathcal{L}_{\psi(T)}$ for some non trivial additive character ψ. Indeed, $\mathcal{L}_{\psi(T)}$ is geometrically irreducible and $\text{sing}(\mathcal{L}_{\psi(T)}) = \{\infty\}$. But we know that $\mathcal{L}_{\psi(T)} \not\cong_{\text{geom}} \mathcal{L}_{\chi(T)}$ for any multiplicative character χ since $\mathcal{L}_{\psi(T)}$ is wild ramified at ∞ while $\mathcal{L}_{\chi(T)}$ is tame everywhere.

Exercise 8. One starts proving the following: let ρ_1 and ρ_2 be two geometrically irreducible ℓ-adic sheaf modulo p, both of which are ramified at most at ∞ with $\mathrm{Swan}_\infty(\rho_1), \mathrm{Swan}_\infty(\rho_2) \leq 1$. Moreover assume that $\rho_1 \not\cong_{\mathrm{geom}} \rho_2$. Then

$$\sum_{x \in \mathbb{F}_p} t_{\rho_1}(x)\overline{t_{\rho_1}(x)} = 0.$$

As usual, one starts with the Euler-Poincaré formula getting

$$h^0 - h^1 + h^2 = \mathrm{rank}(\rho_1 \otimes \rho_2) - \mathrm{Swan}_\infty(\rho_1 \otimes \rho_2).$$

We know from the lecture that $h^0 = 0$ and that $h^2 = 0$ since ρ_1 and ρ_2 are two geometrically irreducible ℓ-adic sheaf modulo p and $\rho_1 \not\cong_{\mathrm{geom}} \rho_2$. Thus

$$-h^1 = \mathrm{rank}(\rho_1 \otimes \rho_2) - \mathrm{Swan}_\infty(\rho_1 \otimes \rho_2).$$

Let us bound $\mathrm{Swan}_\infty(\rho_1 \otimes \rho_2)$. In the lectures we have seen that

$$\mathrm{Swan}_\infty(\rho_1 \otimes \rho_2) \leq \mathrm{rank}(\rho_1 \otimes \rho_2)\lambda_\infty(\rho_1 \otimes \rho_2),$$

where $\lambda_\infty(\rho_1 \otimes \rho_2)$ is the maximal breaks of $\rho_1 \otimes \rho_2$ at ∞. Moreover we know that

$$\lambda_\infty(\rho_1 \otimes \rho_2) \leq \max(\lambda_\infty(\rho_1), \lambda_\infty(\rho_2)).$$

On the other hand

$$\lambda_\infty(\rho_1), \lambda_\infty(\rho_2) \leq 1$$

since $\mathrm{Swan}_\infty(\rho_1), \mathrm{Swan}_\infty(\rho_2) \leq 1$. Hence

$$\mathrm{Swan}_\infty(\rho_1 \otimes \rho_2) \leq \mathrm{rank}(\rho_1 \otimes \rho_2).$$

Inserting this in the Euler-Poincaré formula one gets

$$-h^1 \geq 0,$$

which implies $h^1 = 0$. Hence

$$\sum_{x \in \mathbb{F}_p} t_{\rho_1}(x)\overline{t_{\rho_1}(x)} = 0,$$

as we wanted. To conclude the exercise it is enough to use the same argument as in Exercise 7 replacing the multiplicative characters by the additive ones.