D-MATH
Prof. Paul Nelson

Commutative Algebra

HS 2018

Exercise Sheet 1

Radical ideals, local rings and affine varieties

Important: all exercises in the present sheet assume familiarity with the contents of the first chapter of [1].

Let A be a commutative ring, k an algebraically closed field.

1. (From the lecture) Let $I \subset A$ be a subset such that $AI \subset I$, i.e. $xy \in I$ for all $x \in A$ and $y \in I$. Is it true that I is an ideal?

2. Let $\mathfrak{p} \subset A$ be a proper ideal. Show that the following are equivalent:

   (a) $\mathfrak{p}$ is a prime ideal;

   (b) if $\mathfrak{a}, \mathfrak{b} \subset A$ are ideals in A such that $\mathfrak{ab} \subset \mathfrak{p}$, then either $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

3. Consider the non-commutative ring $\mathcal{M}_n(\mathbb{R})$ of square $n \times n$ matrices with real entries. Find a counterexample to show that the sum of two nilpotent elements need not be nilpotent.

   (Hence, if the ring is not commutative, the set of nilpotent elements is not necessarily an ideal.)

4. Let A be an integral domain with a finite number of elements. Show that A is a field. Deduce that in a finite commutative ring A every prime ideal is maximal.

5. * Let $\mathfrak{a} \subset A$ be an ideal. Show that its radical $r(\mathfrak{a})$ is an ideal. Furthermore, prove that:

   (a) $\mathfrak{a} \subset r(\mathfrak{a})$;

   (b) $r(\mathfrak{a}) = r(r(\mathfrak{a}))$;

   (c) $r(\mathfrak{ab}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$;

   (d) $r(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$;

   (e) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$;

(f) if $\mathfrak{p} \subset A$ is a prime ideal, then $r(\mathfrak{p}^k) = \mathfrak{p}$ for all integer $k > 0$.

6. Prove that every prime ideal is radical. Find a counterexample to show that the converse is not true.

7. Let $n > 0$ be a positive integer. Find the unique positive integer m such that $r((n)) = (m)$.

   (Recall that, given an element $a \in A$, the notation (a) stands for the principal ideal in A generated by a.)

8. * Consider the polynomial ring $A[X]$. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial. Prove that:

   (a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \dots, a_n$ are nilpotent.

   (b) f is nilpotent if and only if $a_0, \dots, a_n$ are nilpotent.

   (c) f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.

9. * Prove that, in the ring $A[X]$, the Jacobson ideal is equal to the nilradical.

10. Let $A[[X]]$ denote the ring of formal power series $f = \sum_{n=0}^{\infty} a_n X^n$ with coefficients in A. Show that:

    (a) f is a unit in $A[[X]]$ if and only if a_0 is a unit in A;

    (b) if f is nilpotent, then a_n is nilpotent for all $n \geqslant 0$. Is the converse also true?

    (c) f belongs to the Jacobson radical of $A[[X]]$ if and only if a_0 belongs to the Jacobson radical of A.

11. * Fix an element $x_0 \in \mathbb{R}^n$. Denote by $\mathcal{U} := \{U \subset \mathbb{R}^n \text{ open} : x_0 \in U\}$ the set of open neighborhoods of x_0, and define the set

    $$S := \{(U, f) : U \in \mathcal{U},\ f\colon U \to \mathbb{R} \text{ continuous}\} \quad .$$

    We define an equivalence relation on S as follows: two elements (U, f), $(V, g) \in S$ are equivalent if there is an open neighborhood $W \subset U \cap V$ of x_0 such that $f|_W = g|_W$. We denote by R the set of equivalence classes. It is called the *ring of germs* of continuous functions. Prove that R is a local ring.

12. * Given a subset $T \subset k^n$, we say that T is an *algebraic set* (or, as in the lecture, a *variety*) if there exists a set $S \subset k[X_1, \dots, X_n]$ such that $T = Z(S)$.

(a) Prove that the set of all algebraic sets in k^n is closed under finite unions and arbitrary intersections. Deduce that the set

$$\tau = \{V \subset k^n : k^n \setminus V \text{ is an algebraic set}\}$$

is the set of open sets for a topology on k^n, which will be henceforth called the *Zariski topology* on k^n.

(b) Describe the Zariski topology on an algebraically closed field k.

(c) Prove that the Zariski topology on $\mathbb{C}^n$ is strictly coarser than the euclidean topology.

13. Let $X \subset k^n$ be a subset. Define

$$I(X) = \{f \in k[X_1, \ldots, X_n] : f(P) = 0 \text{ for all } P \in X\} \quad .$$

Show that $I(X)$ is an ideal in $k[X_1, \ldots, X_n]$ and that it is radical.

14. * Let $X, X' \subset k^n$ and $S, S' \subset k[X_1, \ldots, X_n]$ be subsets. Show the following inclusions (see exercise 13 for notation):

(a) $X \subset Z(S) \Longleftrightarrow S \subset I(X)$;

(b) $Z(S \cup S') = Z(S) \cap Z(S')$;

(c) $I(X \cup X') = I(X) \cap I(X')$;

(d) $S \subset S' \Longrightarrow Z(S) \supset Z(S')$;

(e) $X \subset X' \Longrightarrow I(X) \supset I(X')$;

(f) $S \subset I(Z(S))$ and $X \subset Z(I(X))$;

(g) $Z(S) = Z(I(Z(S)))$ and $I(X) = I(Z(I(X)))$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.