Exercise Sheet 10

Completions of rings and modules

Let R be a commutative ring, k an algebraically closed field.

1. (a) Fix a prime $p \geqslant 2$. For every integer $n \geqslant 1$, define a injective morphism of groups $\alpha_n \colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ by setting $\alpha_n(1) = p^{n-1} \bmod p^n\mathbb{Z}$. Let M denote the direct sum of countably many copies of $\mathbb{Z}/p\mathbb{Z}$, and $N = \bigoplus_{n\geqslant 1} \mathbb{Z}/p^n\mathbb{Z}$. Both M and N are considered naturally as $\mathbb{Z}$-modules in what follows.

   Prove that the p-adic completion of M (i.e. the completion with respect to the descending filtration of submodules $(p^n M)_n$) is M itself, but the completion of M with respect to the topology induced by the p-adic topology on N is the direct *product* $\prod_{n\geqslant 1} \mathbb{Z}/p\mathbb{Z}$ (here M clearly injects into M by means of the direct sum of all maps α_n, hence the induced topology on M is the subspace topology).

   (b) Deduce from the previous point that p-adic completion is *not* a right-exact functor on the category of all $\mathbb{Z}$-modules.

2. Let R be a noetherian ring, $\mathfrak{a} \subset R$ an ideal, $\hat{R}$ the $\mathfrak{a}$-adic completion of R. For any $x \in R$, denote by $\hat{x}$ its image in $\hat{R}$.

   (a) Prove that $\hat{x}$ is not a zero-divisor in $\hat{R}$ whenever x is not a zero-divisor in R.

   (b) Does the previous point imply that $\hat{R}$ is an integral domain provided that R is an integral domain? *(Hint: prove that the completion of R with respect to the product of two coprime ideals is isomorphic to the direct product of the completions with respect to each ideal separately.)*

3. Let R be a noetherian ring, $\mathfrak{a} \subset R$ an ideal. Prove that $\mathfrak{a}$ is contained in the Jacobson radical of R if and only if every maximal ideal of R is closed for the $\mathfrak{a}$-adic topology on R (a noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a *Zariski ring*).

4. Let R be a noetherian ring, $\mathfrak{a} \subset R$ an ideal. Denote by $\hat{R}$ the $\mathfrak{a}$-adic completion of R. Prove that the following two conditions are equivalent:

   (a) R is a Zariski ring (see previous exercise for the terminology);

   (b) for any finitely generated R-module M, the canonical map $M \to \hat{M}$ is injective (where $\hat{M}$ denotes the completion of M with respect to the $\mathfrak{a}$-stable filtration $(\mathfrak{a}^n M)_n$).

(Hint: use the Krull intersection theorem, more precisely its corollary given in Corollary 10.19 *in [1], and the previous exercise.)*

5. The aim of this exercise is to prove another version of *Hensel's lemma*: let R be a local ring with maximal ideal $\mathfrak{m} \subset R$, and assume that R is complete with respect to its $\mathfrak{m}$-adic filtration. For any polynomial $f(X) \in R[X]$, denote by $\bar{f}(X)$ its reduction modulo $\mathfrak{m}$, so that $\bar{f}(X) \in (R/\mathfrak{m})[X]$. Assume that $f(X)$ is monic of degree n and there exist coprime monic polynomials $\bar{g}(X), \bar{h}(X) \in (R/\mathfrak{m})[X]$ of degrees $r, n-r$ respectively with $\bar{f} = \bar{g}\bar{h}$. Then there are monic polynomials $g, h \in R[X]$ such that $\bar{g}, \bar{h}$ are their respective reductions modulo $\mathfrak{m}$ and $f = gh$.

   (a) Assume inductively that we have constructed $g_k, h_k \in R[X]$ with $g_k h_k - f \in \mathfrak{m}^k R[X]$. Use the fact that $\bar{g}$ and $\bar{h}$ are coprime to find $\bar{a}_p, \bar{b}_p \in (R/\mathfrak{m})[X]$ of degree $\leqslant n-r, r$ respectively, such that $X^p = \bar{a}_p \bar{g}_k + \bar{b}_p \bar{h}_k$ in $(R/\mathfrak{m})[X]$, where p is an integer between 1 and n.

   (b) Use completeness of R to show that the sequences $(g_k)_k$ and $(h_k)_k$ converge to some polynomials $g, h \in R[X]$. Prove that g, h thus defined verify the conclusion of Hensel's lemma.

6. Prove the following corollary of Hensel's lemma: let k be a field, $f(T, X)$ a polynomial in two variables with coefficients in k, and assume that $a \in k$ is a simple root of the polynomial $f(0, X) \in k[X]$. Then there exists a unique power series $X(T) \in k[[T]]$ such that $X(0) = a$ and $f(T, X(T)) = 0$ identically in $k[[T]]$.

   (Hint: apply Hensel's lemma as stated in [2], Theorem 7.3, *to* $R = k[[T]]$ *and* $\mathfrak{m} = (T)$.*)*

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.