D-MATH | Commutative Algebra | HS 2018
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Exercise Sheet 11

Completions, projective modules and the Tor functor

Let R be a commutative ring, k an algebraically closed field.

1. Let P be a module over a commutative ring R. Prove that the following conditions are equivalent:

   (a) for any exact sequence $M' \to M \to M''$ of R-modules, the associated sequence
   $$\mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M'')$$
   is exact;

   (b) P is projective;

   (c) any short exact sequence of R-modules of the form $0 \to M' \to M \to P \to 0$ splits (see Exercise 4, Sheet 4 for the definition);

   (d) P is isomorphic to a direct factor (see Exercise 2, Sheet 4 for the definition) of a free R-module.

   (Hint for (d) $\Rightarrow$ (a): prove that the condition expressed in (a) is stable under direct sum, i.e. that a direct sum $\bigoplus_i P_i$ of R-modules satisfies the condition if and only if each factor P_i satisfies it.)

2. Prove the *Snake Lemma*: if $0 \to A \to B \to C \to 0$, $0 \to A' \to B' \to C' \to 0$ are short exact sequences of R-modules, and $\alpha\colon A \to A', \beta\colon B \to B', \gamma\colon C \to C'$ are R-linear maps defining a morphism between the two exact sequences (i.e. such that the resulting diagram commutes), then there is an exact sequence
   $$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0\ .$$

3. (a) Let P be a projective module over a ring R. Show that $\mathrm{Tor}_i^R(M, P) = 0$ for every R-module M and every integer $i > 0$.

   (b) i. Show that an R-module M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for any R-module N.

   ii. Show that an R-module M is flat if and only if $\mathrm{Tor}_i^R(M, N) = 0$ for any R-module N and any integer $i > 0$.

4. Let R be a commutative ring, $x \in R$ a nonzerodivisor. Prove that

$$\mathrm{Tor}_1(R/(x), M) \simeq \{m \in M : xm = 0\}$$

(which incidentally explains the name "Tor", since it is connected with torsion elements in this elementary example).

5. Let R be a local ring with maximal ideal $\mathfrak{m}$, M an R-module. We say that a free resolution

$$F : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \cdots F_0 \to M \to 0$$

of M is *minimal* if each φ_i has image contained in $\mathfrak{m}F_{i-1}$.

If F is a minimal resolution of M as above and $\mathrm{rank} F_i = b_i$ for all $i \in \mathbb{N}$, then prove that

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) \simeq (R/\mathfrak{m})^{b_i} \ .$$

The b_i are called the *Betti numbers* of M, by analogy with the corresponding algebraic topology context, in which F is a chain complex.

6. Let R be a noetherian ring, $\mathfrak{m} \subset R$ an ideal, $\hat{\mathfrak{m}}$ the corresponding ideal in the $\mathfrak{m}$-adic completion $\hat{R}$.

   (a) Prove that $\hat{\mathfrak{m}}$ is contained in the Jacobson radical of $\hat{R}$.

   (b) Deduce from the previous point that, if R is a noetherian local ring and $\mathfrak{m}$ is its maximal ideal, then $\hat{R}$ is a (noetherian) local ring with maximal ideal $\hat{\mathfrak{m}}$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.