D-MATH Commutative Algebra HS 2018
Prof. Paul Nelson

Exercise Sheet 12

Injective modules, Ext functor and Artin rings

Let R be a commutative ring, k an algebraically closed field.

1. Let M be a finitely presented module over a ring R, $\varphi\colon R^n \to M$ a surjective R-linear map. Prove that $\ker\varphi$ is finitely generated.

   (Hint: use Snake Lemma, Exercise 2 Sheet 11.)

2. An R-module Q is called *injective* if, for every monomorphism of R-modules $\alpha\colon N \to M$ and every homomorphism of R-modules $\beta\colon N \to Q$, there exists an homomorphism of R-modules $\gamma\colon M \to Q$ such that $\beta = \gamma \circ \alpha$.

   (a) Prove the following statement: let Q be an R-module, and assume that for every ideal $I \subset R$ and every homomorphism of R-modules $\beta\colon I \to Q$ there is an extension of β to an R-module homomorphism $R \to Q$. Then Q is injective.

   (Hint: use Zorn's lemma to construct the desired extension.)

   (b) Use the previous point to show that an abelian group Q is an injective $\mathbb{Z}$-module if and only if it is *divisible*, i.e. for every $q \in Q$ and every $0 \neq n \in \mathbb{Z}$ there exists $q' \in Q$ such that $nq' = q$.

3. Given an R-module M, an *injective resolution* of M is an exact sequence of R-modules
$$0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots$$
in which the $Q_i, i \geqslant 0$ are injective modules.

   (a) Assuming (without proving it) that every module can be embedded into an injective module, prove that any R-module admits an injective resolution.

   (b) Give an example of an injective resolution of $\mathbb{Z}$ as $\mathbb{Z}$-module.

   (Hint: an immediate consequence of point (b) of the previous exercise is that, if Q is an injective abelian group and K is a subgroup, then Q/K is an injective abelian group.)

4. The purpose of this exercise is to give another example of a derived functor (we already saw the functor Tor, which is left-derived, in the lecture).

   The functor $\mathrm{Hom}_R(M, -)$, where R is a commutative ring and M is a fixed R-module, is left-exact, i.e. it transforms exact sequences of the form $0 \to N' \to$

$N \to N''$ into exact sequences of the same form. If N is an arbitrary R-module, let

$$I : 0 \to N \to I_0 \to I_1 \to \cdots$$

be an injective resolution of N (it always exists by the previous exercise), and form the associated complex

$$\mathrm{Hom}_R(M, I) : \ 0 \to \mathrm{Hom}_R(M, I_0) \to \mathrm{Hom}_R(M, I_1) \to \cdots \quad .$$

We define the *Ext* functor $\mathrm{Ext}^i_R(M, N)$ to be the homology module $H_{-i}(\mathrm{Hom}_R(M, I))$, for all integer $i \geqslant 0$. As in the case of the Tor functor, it can be shown that the definition does not depend on the choice of the injective resolution of N.

(a) Let $x \in R$ be a nonzerodivisor. For any R-module M, compute $\mathrm{Ext}^i_R(R/(x), M)$. In particular, compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for any integers n, m.

(b) Prove that a finitely generated abelian group A is free (as a $\mathbb{Z}$-module) if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$.

5. Let R be a noetherian ring. Prove that the following are equivalent:

   (a) R is an Artin ring;

   (b) $\mathrm{Spec}(R)$ is discrete (w.r.t. the Zariski topology) and finite;

   (c) $\mathrm{Spec}(R)$ is discrete (w.r.t. the Zariski topology).

6. Let k be a field, and consider the ring $R = k[X^2, X^3]/(X^n)$, where n is a sufficiently large integer (e.g. $n \geqslant 10$). Prove that R has just one prime ideal and conclude that it is zero-dimensional.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.