Exercise Sheet 13

Dimension theory, valuation rings and Dedekind domains

1. Let R be a commutative ring. Show that the n-th symbolic power $\mathfrak{p}^{(n)}$ of a prime ideal $\mathfrak{p} \subset R$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$ (here $n \geqslant 1$ is an integer).

2. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $k := A/\mathfrak{m}$ denote its residue field. Let $f_1, \ldots, f_r \in \mathfrak{m}$. Set $\overline{A} := A/(f_1, \ldots, f_r)$. Let $\overline{\mathfrak{m}} \subset \overline{A}$ denote the image of $\mathfrak{m}$.

   (a) Show that $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) \geqslant \dim(\overline{A}) \geqslant d - r$.

   (b) Assume that A is regular. Let $\overline{f_1}, \ldots, \overline{f_r} \in \mathfrak{m}/\mathfrak{m}^2$ denote the images of $f_1, \ldots, f_r$. Show that the following are equivalent:

      i. $\overline{A}$ is regular of dimension $d - r$.

      ii. $\overline{f_1}, \ldots, \overline{f_r}$ are linearly independent over k.

3. Let k be a field, and denote by $R = k[[X_1, \ldots, X_n]]$ the ring of formal power series in n variables with coefficients in k. Prove that R is a regular, noetherian local ring of dimension n.

4. Let R be a Dedekind domain, $\mathfrak{a} \neq 0$ an ideal in R.

   (a) Show that every ideal in $R/\mathfrak{a}$ is principal.

   (b) Deduce that any ideal in R can be generated by at most 2 elements.

5. Let G be a totally ordered abelian group, k a field. Denote by R the vector space over k with basis $(e_\alpha)_{0 \leqslant \alpha \in G}$ (i.e, the vector space of formal k-linear combinations of the elements e_α, $0 \leqslant \alpha \in G$). Define a product between basis elements by means of the formula $e_\alpha \cdot e_\beta = e_{\alpha+\beta}$, for every $0 \leqslant \alpha, \beta \in G$, and extend it by k-linearity to the whole of R. Prove that this operation makes R into a valuation ring with value group G and valuation
$$v\left(\sum_\alpha r_\alpha e_\alpha\right) = \min\{\alpha \in G : r_\alpha \neq 0\} ,$$
where the r_α are elements of the field k.

6. Let R be an integral domain. Prove the following statements:

   (a) R is a valuation ring if and only if, for every pair of ideals $\mathfrak{a}, \mathfrak{b} \subset R$, we have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$;

   (b) if R is a valuation ring and $\mathfrak{p} \subset R$ is a prime ideal, then $R_\mathfrak{p}$ and $R/\mathfrak{p}$ are both valuation rings.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.