D-MATH
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Commutative Algebra

HS 2018

Exercise Sheet 3

Spectrum of a ring, modules and primary decomposition

Let R be a commutative ring, k an algebraically closed field.

1. The *spectrum* of a commutative ring R is defined as the set

$$\mathrm{Spec}(R) = \{P \subset R : P \text{ is a prime ideal}\}$$

The purpose of this exercise is to show that $\mathrm{Spec}(R)$ can be equipped with a topology, called the *Zariski topology*, making it into a compact topological space.

Define a subset $X \subset \mathrm{Spec}(R)$ to be *closed* if it is empty or else if there exists an ideal $\mathfrak{a} \subset R$ such that

$$X = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{a} \subset \mathfrak{p}\} .$$

(a) Prove that, if $X_1, X_2 \subset \mathrm{Spec}(R)$ are closed, then so is $X_1 \cup X_2$.

(b) Prove that, if $(X_i)_i$ is a collection of closed subsets of $\mathrm{Spec}(R)$, then so is $\bigcap_{i\in I} X_i$.

(c) Deduce, from the previous two points, that the complements in $\mathrm{Spec}(R)$ of closed subsets are the open sets for a topology on $\mathrm{Spec}(R)$.

(d) Let $(X_i)_{i\in I}$ be a collection of closed subsets of $\mathrm{Spec}(R)$ with the finite intersection property, namely such that $\bigcap_{j\in J} X_j \neq \varnothing$ for any finite subset $J \subset I$. Show that this implies $\bigcap_{i\in I} X_i \neq \varnothing$.

Deduce that $\mathrm{Spec}(R)$, with the topology defined in (c), is a compact topological space.

(e) Which condition should an ideal $\mathfrak{p} \subset R$ satisfy for the singleton $\{\mathfrak{p}\} \subset \mathrm{Spec}(R)$ to be closed?

(f) Show that $\mathrm{Spec}(R)$ is a T_0-space, i.e. for any two distinct points $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(R)$ either there exists a neighborhood of $\mathfrak{p}_1$ not containing $\mathfrak{p}_2$ or there exists a neighborhood of $\mathfrak{p}_2$ not containing $\mathfrak{p}_1$.

(g) Is $\mathrm{Spec}(R)$ always a Hausdorff topological space?

2. Let A, B be two commutative rings, $\varphi\colon A \to B$ a ring homomorphism. For any ideal $\mathfrak{b} \subset B$, denote the ideal $\varphi^{-1}(\mathfrak{b})$ by $\varphi^*(\mathfrak{b})$.

(a) Show that the assignment $\mathrm{Spec}(B) \ni \mathfrak{p} \mapsto \varphi^*(\mathfrak{p})$ gives a well-defined map $\varphi^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

(b) Prove that φ^* is continuous, where both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are equipped with the Zariski topology.

(c) Let $\psi\colon B \to C$ be a ring homomorphism. Show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. (Hence, in the language of categories, the assignment $R \mapsto \mathrm{Spec}(R)$ defines a *contravariant functor* from the category of commutative rings to the category of topological spaces).

(d) Assume that φ is surjective. Prove that φ^* is an homeomorphism of $\mathrm{Spec}(B)$ onto the closed subset $X_0 = \{\mathfrak{p} \in \mathrm{Spec}(A) : \ker\varphi \subset \mathfrak{p}\}$ of $\mathrm{Spec}(A)$.

(e) Deduce from the previous point that, for an arbitrary commutative ring A, $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathrm{nil}(A))$ are naturally homeomorphic, where $\mathrm{nil}(A)$ denotes the nilradical ideal of A.

(f) Assume that φ is injective. Prove that $\varphi^*(\mathrm{Spec}(B))$ is dense in $\mathrm{Spec}(A)$. More precisely, show that $\varphi^*(\mathrm{Spec}(B))$ is dense in $\mathrm{Spec}(A)$ if and only if $\ker\varphi \subset \mathrm{nil}(A)$.

3. In this exercise we examine a connection between the Zariski topology on the spectrum of a ring and the Zariski topology on the affine space k^n (k algebraically closed field). Let $X \subset k^n$ be a variety, and denote by $I(X)$ the ideal of $k[X_1, \ldots, X_n]$ defined by it. The quotient ring

$$P(X) = k[X_1, \ldots, X_n]/I$$

is called the *(affine) coordinate ring* of X.

(a) Define $\tilde{P}(X)$ to be the *ring of polynomial functions* on X, namely

$$\tilde{P}(X) = \{\varphi\colon X \to k : \exists\, f \in k[X_1, \ldots, X_n] \text{ s.t. } \varphi(x) = f(x)\ \forall\, x \in X\}\,,$$

with the obvious addition and multiplication operations. Show that $P(X)$ and $\tilde{P}(X)$ are isomorphic rings.

(b) For each $x \in X$, denote by $\mathfrak{m}_x$ the ideal of all $f \in P(X)$ such that $f(x) = 0$. Show that it is a maximal ideal in $P(X)$.

Hint: in the one-to-one correspondence between affine varieties in k^n and radical ideals of $k[X_1, \ldots, X_n]$, prove that, for any variety $X \subset k^n$, $I(X)$ is a maximal ideal whenever $X = \{P\}$ is a singleton.

(c) Given an arbitrary commutative ring R, we define

$$\mathrm{Max}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \text{ is a maximal ideal}\}\,;$$

$\mathrm{Max}(R)$ is called the *maximal spectrum* of R.

In the previous point, we have thus defined a map $\mu\colon X \to \mathrm{Max}(P(X))$. Prove that μ is injective.

(d) Using the weak form of the Hilbert Nullstellensatz, prove that the map μ is surjective.

(e) Suppose now that $X = k^n$, so that $P(X) \simeq k[X_1, \dots, X_n]$. Show that the map μ is continuous with respect to the Zariski topologies on k^n and on $\mathrm{Max}(k[X_1, \dots, X_n])$. Is μ an homeomorphism onto its image?

4. Let $(M_i)_{i \in I}$ be a collection of modules over the commutative ring R. Denote by $\bigoplus_{i \in I} M_i$ their direct sum, and by $\prod_{i \in I} M_i$ their product. For any $j \in I$, denote by $\pi_j \colon \prod_{i \in I} M_i \to M_j$ the canonical (linear) projection onto the j-th factor, and by $\eta_j \colon M_j \to \bigoplus_{i \in I} M_i$ the injective linear map defined by

$$(\eta_j(x))_i = \begin{cases} x & \text{if } i = j \ ; \\ 0 & \text{if } i \neq j \ . \end{cases}$$

(a) Prove the *universal property of the product*: for any R-module N and any collection $(\varphi_i)_{i \in I}$ of linear maps $\varphi_i \colon N \to M_i$, there exists a unique linear map $\varphi \colon N \to \prod\limits_{i \in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for all $j \in I$.

(b) Prove the *universal property of the direct sum*: for any R-module P and any collection $(\psi_i)_{i \in I}$ of linear maps $\psi_i \colon M_i \to P$, there exists a unique linear map $\psi \colon \bigoplus_{i \in I} M_i \to P$ such that $\psi \circ \eta_j = \psi_j$ for all $j \in I$.

(c) Let $(N_j)_{j \in J}$ be another collection of R-modules. Use the previous two points to prove that there exists a canonical linear isomorphism

$$\mathrm{Hom}\left(\bigoplus_{i \in I} M_i, \prod_{j \in J} N_j \right) \to \prod_{(i,j) \in I \times J} \mathrm{Hom}(M_i, N_j) \ .$$

5. In this exercise we discuss the notion of *direct limits of modules.*

Let $(I, \leqslant)$ be a directed set, i.e. a partially ordered set with the property that for all $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Let $(M_i)_{i \in I}$ be a collection of R-modules, and assume that we are given, for each $i \leqslant j \in I$, an R-module morphism $\mu_{ij} \colon M_i \to M_j$ such that:

- $\mu_{ii} \colon M_i \to M_i$ is the identity map for all $i \in I$;
- $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ for all $i \leqslant j \leqslant k \in I$.

The modules $(M_i)_{i \in I}$ together with the collection $(\mu_{ij})_{i \leqslant j \in I}$ form a so-called *direct system* of modules.

Denote by C the direct sum of all modules $M_i, i \in I$ and identify each factor M_i with its isomorphic image in C. Let D be the submodule of C generated by the set $\{x_i - \mu_{ij}(x_i) : x_i \in M_i, i \leqslant j \in I\}$. Let $M = C/D$, and let $\mu \colon C \to M$ be the canonical projection map. Denote by μ_i the restriction of μ to M_i The module M is called the *direct limit* of the direct system, and it is denoted by $M = \varinjlim M_i$.

(a) Prove that $\mu_i = \mu_j \circ \mu_{ij}$ for all $i \leqslant j \in I$.

(b) Show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

(c) Show that, if $\mu_i(x_i) = 0$ for some $x_i \in M_i$ and some $i \in I$, then there exists $j \geqslant i$ such that $\mu_{ij}(x_i) = 0$.

(d) Prove the *universal property of the direct limit*: for any R-module N and any collection $(\varphi_i)_{i\in I}$ of R-linear maps $\varphi_i \colon M_i \to N$ such that $\varphi_i = \varphi_j \circ \mu_{ij}$ for all $i \leqslant j \in I$, there exists a unique R-linear map $\varphi \colon M \to N$ such that $\varphi_i = \varphi \circ \mu_i$ for all $i \in I$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.