Exercise Sheet 6

Associated primes and primary decomposition

Let R be a commutative ring, k an algebraically closed field.

1. Let k be a field. A *monomial ideal* is an ideal $I \subset k[X_1, \ldots, X_n]$ generated by monomials in the variables $X_1, \ldots, X_n$.
   (a) Characterize those monomial ideals which are prime in $k[X_1, \ldots, X_n]$.
   (b) Which monomial ideals are irreducible? Radical? Primary? (Recall that a submodule of a module is called *irreducible* if it cannot be written as the intersection of two larger submodules)

2. The setting is the same as in Exercise 1.
   (a) Find an algorithm to compute the radical of a monomial ideal.
   (b) Find an algorithm to compute an irreducible decomposition, and thus a primary decomposition, of a monomial ideal.

3. The setting is again the same as in the previous two exercises.
   (a) Let I be the product ideal of the ideals $(X_1), (X_1, X_2), \ldots, (X_1, \ldots, X_n)$. Determine the associated primes of I.
   (b) More generally, for any subset $J \subset \{1, \ldots, n\}$, let $P(J)$ be the prime ideal generated by $\{X_j, j \in J\}$. Let $I_1, \ldots, I_t$ be subsets of $\{1, \ldots, n\}$, and set $I = \prod_{i=1}^{t} P(I_i)$. Let Γ be the "incidence graph", whose vertices are the set $I_i, i = 1, \ldots, t$, with an edge joining I_i and I_j if and only if $I_i \cap I_j \neq \varnothing$. Show that the associated primes of I are precisely those primes that may be expressed as $P(I_{j_1} \cup \cdots \cup I_{j_s})$, where $I_{j_1}, \ldots, I_{j_s}$ forms a connected (i.e., any two vertices can be joined by a finite path made up of edges) subgraph of Γ.

4. (On the uniqueness of primary decomposition) Let $R = k[X, Y]/I$, where k is a field and $I = (X^2, XY) = (X) \cap (X, Y)^2$. Show that the ideal (Y^n) is (X, Y)-primary in R (considered as a module over $k[X, Y]$), and that

$$0 = (X) \cap (Y^n)$$

   is a minimal primary decomposition of 0 in R for any integer $n \geqslant 1$.

5. Let R be an arbitrary commutative ring, $\mathfrak{a} \subset R$ an ideal.

(a) Suppose that its radical $\mathrm{Rad}(\mathfrak{a})$ is a maximal ideal. Show that $\mathfrak{a}$ is a primary ideal.

(b) Deduce from the previous point that arbitrary powers of maximal ideals are primary ideals.

6. Let X be an infinite compact Hausdorff space. Consider the ring $C(X)$ of real-valued continuous functions on X. Is the zero ideal decomposable in this ring?

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.