D-MATH Commutative Algebra HS 2018
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Exercise Sheet 7

Primary decomposition and integrality

Let R be a commutative ring, k an algebraically closed field.

1. Let X be a topological space.

   (a) Assume that X is irreducible. Show that any non-empty open subset $O \subset X$ is dense in X and an irreducible topological space when endowed with the subspace topology.

   (b) Assume that $Y \subset X$ is irreducible as a subspace. Show that the closure $\overline{Y}$ in X is also irreducible.

   (c) Show that any irreducible subspace of X is contained in a maximal irreducible subspace.

   (d) Prove that the maximal irreducible subspaces are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

   (e) Let $X = \operatorname{Spec}(R)$, where R is a commutative ring. Prove that the irreducible components of X are the closed sets $V(\mathfrak{p}) = \{\mathfrak{p}' \in \operatorname{Spec}(R) : \mathfrak{p}' \supset \mathfrak{p}\}$, where $\mathfrak{p}$ is a minimal prime ideal of R.

2. Let R be a commutative ring, and denote by $R[X]$ the ring of polynomials in one indeterminate over R. For any ideal $\mathfrak{a} \subset R$, denote by $\mathfrak{a}[X]$ the set of all polynomials in $R[X]$ with coefficients in $\mathfrak{a}$.

   (a) Prove that $\mathfrak{a}[X]$ is the extension of the ideal $\mathfrak{a}$ in $R[X]$.

   (b) Prove that, if $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p}[X]$ is a prime ideal in $R[X]$.

   (c) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in R, then show that $\mathfrak{q}[X]$ is $\mathfrak{p}$-primary in $R[X]$.

   (d) If $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a minimal primary decomposition of $\mathfrak{a}$ in R, then show that $\mathfrak{a}[X] = \bigcap_{i=1}^{n} \mathfrak{q}_i[X]$ is a minimal primary decomposition of $\mathfrak{a}[X]$ in $R[X]$.

   (e) Prove that, if $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$ in R, then $\mathfrak{p}[X]$ is a minimal prime ideal of $\mathfrak{a}[X]$ in $R[X]$.

3. The purpose of this exercise is to prove the *Krull intersection theorem*: let R be a noetherian ring, $\mathfrak{a} \subset R$ an ideal, and denote by $\mathfrak{b}$ the intersection of all powers $\mathfrak{a}^n, n \geqslant 1$. Then:

   - $\mathfrak{b}\mathfrak{a} = \mathfrak{b}$;

- $\mathfrak{b}(1-a) = (0)$ for some element $a \in \mathfrak{a}$;
- if R is a domain and $\mathfrak{a}$ is a proper ideal, then $\mathfrak{b} = (0)$.

(a) Prove the following preliminary result: if R is a noetherian ring and $I \subset R$ is an ideal, then $(\mathrm{rad}(I))^n \subset I$ for some positive integer n.

(b) Use the previous point to prove the first assertion of Krull's intersection theorem.

(Hint: clearly the non-trivial inclusion is $\mathfrak{b} \subset \mathfrak{b}\mathfrak{a}$. Prove that $\mathfrak{b} \subset \mathfrak{q}$ whenever $\mathfrak{q}$ is a primary ideal containing $\mathfrak{b}\mathfrak{a}$ and deduce the result by means of the primary decomposition theorem)

(c) You may admit the following result: if M is a finitely generated R-module, and $I \subset R$ is an ideal such that $IM = M$, then M is annihilated by an element of the form $1-a$, with $a \in I$.

Using this, prove the last two assertions of Krull's intersection theorem.

4. Let $\varphi\colon R \to S$ be an integral homomorphism of rings. Show that the induced continuous map $\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed, namely that it maps closed sets to closed sets.

5. Let R be a subring of a ring S such that S is integral over R, and let $\varphi\colon R \to \Omega$ be a ring homomorphism of A into an algebraically closed field Ω. Show that φ admits an extension to a ring homomorphism $\Phi\colon S \to \Omega$.

   (Hint: use theorem 5.10 in [1])

6. Let G be a finite group of automorphisms of a ring R, and let R^G denote the subset of G-invariant elements, i.e. $R^G = \{x \in R : \sigma(x) = x \text{ for all } \sigma \in G\}$.

(a) Prove that R^G is a subring of R and that R is integral over R^G.

(b) Suppose $S \subset R$ is a multiplicatively closed subset such that $\sigma(S) \subset S$ for all $\sigma \in G$, and denote by $S^G = S \cap R^G$. Show that the action of G on R extends to an action on $R[S^{-1}]$, and prove that

$$R^G[(S^G)^{-1}] \simeq (R[S^{-1}])^G \ .$$

(c) Let $\mathfrak{p}$ be a prime ideal of R^G, and let P the set of prime ideals of R whose contraction is $\mathfrak{p}$. Show that G acts transitively on P and deduce that P is a finite set.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.