D-MATH | Commutative Algebra | HS 2018
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Exercise Sheet 8

Integrality

Let R be a commutative ring, k an algebraically closed field.

1. Let R be a subring of a ring S such that S is integral over R. Let $\mathfrak{b}$ be a maximal ideal of S and set $\mathfrak{a} = \mathfrak{b} \cap R$, the corresponding maximal ideal of R. Find a counterexample to show that the localization $S_{\mathfrak{b}}$ is not necessarily integral over $R_{\mathfrak{a}}$.

   (Hint: consider the subring $k[X^2-1]$ of $k[X]$, where k is a field, and let $\mathfrak{b} = (X-1)$. Can the element $1/(x+1)$ be integral?)

2. Let R, S be rings, with R a subring of S and S integral over R.
   (a) Prove that, if $x \in R$ is a unit in S, then it is also a unit in R.
   (b) Show that the Jacobson radical of R is the contraction of the Jacobson radical of S.
   (Hint: use 5.8 and 5.10 of [1].)

3. (a) Let R be a subring of an integral domain S, and let S' denote the integral closure of R inside S. Let $f, g \in S[X]$ be monic polynomials with $fg \in S'[X]$. Prove that both f and g belong to $S'[X]$.
   (Hint: consider a field containing S over which f and g split into linear factors.)
   (b) Prove the same result of the previous point without the assumption that S is an integral domain.

4. Let R be a subring of a ring S, and denote by S' the integral closure of R inside S. Prove that the subring $S'[X]$ is the integral closure of $R[X]$ inside $S[X]$.

   (Hint: apply Exercise 3 of the current sheet. You may also look at the hint in [1], Exercise 9 Chapter 5.)

5. Let R be an integral domain containing a subring S isomorphic to $k[T]$ for some field k. Show that, if R is finitely generated as an S-module, then R is free as an S-module.

   (*Hint: use the classification theorem for finitely generated modules over a principal ideal domain, together with the assumption that R is an integral domain.*)

6. The setting is the same as in the previous exercise.

(a) Assume that $R = k[X, Y]/(X^2 - Y^3)$, and let $T = X^m Y^n$ for some integers $m, n \geqslant 1$. Show by exhibiting a basis that the rank of the free S-module R is $3m + 2n$.

(b) Let R, S be again as in the assumptions of Exercise 5. Denote by $\bar{R}$ the integral closure of R. Apply theorem 4.14 of [2] to show that $\bar{R}$ is again finitely generated over S (hence a free S-module), and prove that it has the same rank as R.

(*Hint: localize R and $\bar{R}$ with respect to the prime ideal $\{0\} \subset R$, and use Proposition* 4.13 *of [2]. What is the integral closure of a field?*)

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.