Exercise Sheet 9

Integrality, Jacobson rings and Krull topology

Let R be a commutative ring, k an algebraically closed field.

1. let R, S be commutative rings. A ring homomorphism $f\colon R \to S$ is said to have the *going-up property* if the conclusion of the going-up theorem 5.11 in [1] holds for S and its subring $f(R)$.

   Let $f^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the mapping associated with f.

   Consider the following three statements:

   (a) f^* is a closed mapping;

   (b) f has the going-up property;

   (c) let $\mathfrak{q}$ be any prime ideal of S and $\mathfrak{p}$ the contraction of $\mathfrak{q}$ in R. Then $f^*\colon \mathrm{Spec}(S/\mathfrak{q}) \to \mathrm{Spec}(R/\mathfrak{p})$ is surjective.

   Prove that (a) implies (b), and that (b) is equivalent to (c).

2. The aim of this exercise is to prove *Noether's normalization lemma*: let k be an infinite field, $A \neq 0$ a finitely generated k-algebra. Then there exist elements $y_1, \ldots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \ldots, y_r]$.

   (a) Prove that there are generators $x_1, \ldots, x_n$ of A as a k-algebra such that $x_1, \ldots, x_r$ are algebraically independent over k and each of $x_{r+1}, \ldots, x_n$ is algebraic over $k[x_1, \ldots, x_r]$, for some $1 \leqslant r \leqslant n$.

   (b) Argue by induction on n (if $n = r$ there is nothing to do). Suppose $n > r$ and the result true for $n-1$ generators.

      i. Show that there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \ldots, x_{n-1}, x_n) = 0$.

      ii. Let F be the homogeneous part of highest degree of f. Use the assumption that k is infinite to show that there exist $\lambda_1, \ldots, \lambda_{n-1} \in k$ with $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$.

      iii. Set $x'_i = x_i - \lambda_i x_n$ for all $1 \leqslant i \leqslant n-1$. Prove that x_n is integral over the ring $A' = k[x'_1, \ldots, x'_{n-1}]$, and conclude that A is integral over A'.

   (c) Apply the inductive hypothesis to conclude the proof.

3. Let k be an algebraically closed field.

   (a) Prove that k is infinite.

   (b) Let X be an affine algebraic variety in k^n with coordinate ring $A \neq 0$. Use the outlined proof of Noether's normalization lemma to prove that there exists a linear subspace L of dimension r in k^n and a linear mapping $k^n \to L$ which maps X onto L.

4. Let R be a ring. Show that the following are equivalent:

   (a) R is a Jacobson ring;

   (b) for any ring S and any surjective ring homomorphism $f\colon R \to S$, the nilradical ideal of S coincides with its Jacobson ideal;

   (c) every prime ideal in R which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

   (*Hint: for the implication* (*c*) $\Rightarrow$ (*b*) *argue as follows. Assume* (*b*) *is false, then there is a prime ideal which is not the intersection of maximal ideals. Passing to the quotient ring, we may assume that R is a domain with non-zero Jacobson ideal. Pick a non-zero* f *in the Jacobson ideal, then* $R_f \neq 0$*, thus* R_f *has a maximal ideal whose contraction in R is a prime ideal not containing* f*, and which is maximal with repsect to this property.*)

5. Let G be a group (not necessarily abelian), and let $\mathcal{F}$ be a filter on G satisfying the following properties:

   - for any $V \in \mathcal{F}$ there exists $U \in \mathcal{F}$ such that $UU^{-1} \subset V$, where $UU^{-1} = \{xy^{-1} : x, y \in U\}$;
   - for any $V \in \mathcal{F}$ and any $g \in G$, $gVg^{-1} \in \mathcal{F}$, where $gVg^{-1} = \{gxg^{-1} : x \in V\}$.

   Prove that there exists a unique topology τ on G making G into a topological group (i.e. multiplication and inverse are continuous maps with respect to τ) and for which $\mathcal{F}$ is the filter of neighborhoods of the identity element e_G.

6. Let G be an abelian group, and let $G = G_0 \supset G_1 \supset \cdots$ be a descending filtration of subgroups. Consider the set $\mathcal{F}\{V \subset G : G_n \subset V \text{ for some } n \in \mathbb{N}\}$

   (a) Prove that $\mathcal{F}$ satisfies the conditions in the previous exercise. The resulting topology on G is called the *Krull topology* (determined by the given filtration).

   (b) Show that if $H \in \mathcal{F}$ is a subgroup, then it is open for the Krull topoogy.

   (c) Assume that $G = R$ is a commutative ring, and suppose the G_i are ideals of R. Prove that ring multiplication is continuous with respect to the Krull topology (so that R becomes a *topological ring*).

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.