D-MATH | Commutative Algebra | HS 2018

Prof. Paul Nelson

Exercise Sheet 2

Localization and the Hilbert Nullstellensatz

Let A be a commutative ring, k an algebraically closed field.

1. Let A, B be two commutative rings, $f\colon A \to B$ a ring homomorphism.

   (a) Show that, for every ideal $\mathfrak{b} \subset B$, the preimage $f^{-1}(\mathfrak{b})$ is an ideal in A (it is called the *contraction* of the ideal $\mathfrak{b}$, and it is denoted by $\mathfrak{b}^c$).

   (b) Show that the image $f(\mathfrak{a})$ under f of an ideal $\mathfrak{a} \subset A$ need not be an ideal in B. (For any ideal $\mathfrak{a} \subset A$, the ideal $\mathfrak{a}^e$ of B generated by $f(\mathfrak{a})$ is called the *extension* of the ideal $\mathfrak{a}$.)

   (c) Show that the contraction of a prime ideal is always a prime ideal, while the extension of a prime ideal need not be a prime ideal.

   (d) Prove the following inclusions:

      i. $\mathfrak{a} \subset \mathfrak{a}^{ec}$ for every ideal $\mathfrak{a} \subset A$;

      ii. $\mathfrak{b} \supset \mathfrak{b}^{ce}$ for every ideal $\mathfrak{b} \subset B$;

      iii. $\mathfrak{a}^e = \mathfrak{a}^{ece}$ and $\mathfrak{b}^c = \mathfrak{b}^{cec}$ for any ideals $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$.

   (e) Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset A$, $\mathfrak{b}_1, \mathfrak{b}_2 \subset B$ be ideals. Prove that:

      i. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$, $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c + \mathfrak{b}_2^c$;

      ii. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$, $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$;

      iii. $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$, $(\mathfrak{b}_1\mathfrak{b}_2)^c \supset \mathfrak{b}_1^c\mathfrak{b}_2^c$;

      iv. $r(\mathfrak{a}_1)^e \subset r(\mathfrak{a}_1^e)$, $r(\mathfrak{b}_1)^c = r(\mathfrak{b}_1^c)$.

2. In this exercise, we shall examine more closely the particular case of Exercise 1 in which $B = A[S^{-1}]$, the ring of fractions of A with denominators in a multiplicative subset $S \subset A$, and $f\colon A \to A[S^{-1}]$ is the canonical ring homomorphism.

(a) Let $\mathfrak{a} \subset A$ be an ideal. Show that $\mathfrak{a}^e = \mathfrak{a}[S^{-1}]$, where the latter is defined by

$$\mathfrak{a}[S^{-1}] = \{as^{-1} : a \in \mathfrak{a},\ s \in S\} \quad .$$

(b) For any ideal $\mathfrak{b} \subset A[S^{-1}]$, show that there exists an ideal $\mathfrak{a} \subset A$ such that $\mathfrak{a}^e = \mathfrak{b}$. Deduce that if $\mathfrak{b}_1 \subsetneq \mathfrak{b}_2$ are ideals in $A[S^{-1}]$, then $\mathfrak{b}_1^c \subsetneq \mathfrak{b}_2^c$.

(c) For any ideal $\mathfrak{a} \subset A$, show that equality $\mathfrak{a}^e = A[S^{-1}]$ holds if and only if $\mathfrak{a} \cap S \neq \varnothing$.

(d) Prove that the map $\mathfrak{p} \mapsto \mathfrak{p}[S^{-1}]$ defines a one-to-one correspondence between prime ideals $\mathfrak{p}$ in A such that $\mathfrak{p} \cap S = \varnothing$ and prime ideals in $A[S^{-1}]$.

Deduce that, if $\mathcal{R}$ denotes the nilradical ideal of A, then the nilradical ideal of $A[S^{-1}]$ is $\mathcal{R}[S^{-1}]$.

(e) Assume that $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal. Deduce from the previous point that the prime ideals of the local ring $A_{\mathfrak{p}}$ are in one-to-one correspondence with the prime ideals of A contained in $\mathfrak{p}$.

3. Let A be a commutative ring. Prove that the following two assertions are equivalent:

   (a) A is a noetherian ring.

   (b) A satisfies the *ascending chain condition on ideals*, i.e. for all ascending sequence $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n \subset \cdots$ of ideals of A, there exists an integer $m \geqslant 1$ such that $\mathfrak{a}_m = \mathfrak{a}_{m+j}$ for all integer $j \geqslant 0$.

   Use this characterization of noetherian rings and Exercise 2(b) (of the current sheet) to show that if A is noetherian, then so is $A[S^{-1}]$ for any multiplicative subset $S \subset A$.

4. Let $R = A[X]$ be the ring of polynomials in one variable over the commutative ring A. Consider the multiplicative subset $S = \{1, X, X^2, \dots, X^m, \dots\}$. The ring of fractions $R[S^{-1}]$ is called the *ring of Laurent polynomials* over A.

   Let $A^{(\mathbb{Z})} = \{f \colon \mathbb{Z} \to A : f(n) = 0$ for all but a finite number of $n\}$, which is a commutative ring when endowed with pointwise addition of functions and multiplication given by

$$(f \cdot g)(n) = \sum_{i+j=n} f(i)g(j) \text{ for all } n \in \mathbb{Z}\ , \text{ for all } f, g \in A^{(\mathbb{Z})}\ .$$

   Prove that $R[S^{-1}]$ and $A^{(\mathbb{Z})}$ are isomorphic as rings.

5. Let A be a commutative ring, and assume that for any prime ideal $\mathfrak{p} \subset A$ the localization $A_{\mathfrak{p}}$ at $\mathfrak{p}$ is an integral domain. Is it true then that A is an integral domain?

6. Let A be a commutative ring. Denote by S_0 the set of all non-zero divisors of A.

   (a) Show that S_0 is a multiplicative subset of A. (The ring $A[S_0^{-1}]$ is called the *total ring of fractions* of A).

   (b) Prove that S_0 is the largest multiplicative subset S of A such that the canonical map $A \to A[S^{-1}]$ is injective.

   (c) Show that, in the ring $A[S_0^{-1}]$, each element is either invertible or a zero-divisor.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.