Solutions Sheet 11

Completions, projective modules and the Tor functor

Let R be a commutative ring, k an algebraically closed field.

1. Let P be a module over a commutative ring R. Prove that the following conditions are equivalent:

   (a) for any exact sequence $M' \to M \to M''$ of R-modules, the associated sequence
   $$\mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M'')$$
   is exact;

   (b) P is projective;

   (c) any short exact sequence of R-modules of the form $0 \to M' \to M \to P \to 0$ splits (see Exercise 4, Sheet 4 for the definition);

   (d) P is isomorphic to a direct factor (see Exercise 2, Sheet 4 for the definition) of a free R-module.

   (Hint for $(d) \Rightarrow (a)$: prove that the condition expressed in (a) is stable under direct sum, i.e. that a direct sum $\bigoplus_i P_i$ of R-modules satisfies the condition if and only if each factor P_i satisfies it.)

   Solution. $[(a) \Rightarrow (b)]$ Let $f\colon M \to M''$ be a surjective map of R-modules. This gives a short exact sequence $0 \to M' \to M \to M'' \to 0$, with $M' = \ker f$. BY assumption, the sequence $0 \to \mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M'') \to 0$ is exact. In particular, the map $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M'')$ is surjective. This means that every R-linear map $\varphi\colon P \to M''$ can be *lifted* to an R-linear map $\tilde{\varphi}\colon P \to M$, i.e. a map satisfying $\varphi = f \circ \tilde{\varphi}$. Therefore, P is a projective module.

   $[(b) \Rightarrow (c)]$ Let $0 \to M' \to M \to P \to 0$ be a short exact sequence of R-modules. Thus P is isomorphic to M/M' (more precisely, to the quotient of M by a submodule isomorphic to M', which we will canonically identify with M' itself). Denote by $\varphi\colon P \to M/M'$ this isomorphism. Now, since P is projective and the canonical projection map $\pi\colon M \to M/M'$ is surjective, there exists an R-linear map $s\colon P \to M$ such that $\pi \circ s = \varphi$. This implies that $f \circ s = \mathrm{id}_P$, where $f\colon M \to P$ is the map appearing in the original short exact sequence. Thus, the sequence admits a section, whence it splits.

   $[(c) \Rightarrow (d)]$ Choose a short exact sequence $0 \to N \to F \to P \to 0$ of R-modules, where F is a free R-module (this can always be done for any module P, without the projective assumption). The hypothesis implies that such a sequence splits. Exercise 4, Sheet 4 now gives that P is isomorphic to a direct factor of F. The conclusion is achieved.

$[(d) \Rightarrow (a)]$ We first claim that if $P = F$ is a free R-module, then the condition expressed in (a) holds. The claim is obviously true if $F = R$, by the canonical isomorphism $\mathrm{Hom}(R, N) \simeq N$ for any R-module N.

Our aim will be now to prove the following assertion: if $(P_i)_{i\in I}$ is an arbitrary collection of R-modules then their direct sum $P = \bigoplus_{i\in I} P_i$ satisfies (a) if and only if each P_i satisfies it. Indeed, we have

$$\mathrm{Hom}(P, N) = \mathrm{Hom}\left(\bigoplus_{i\in I} P_i, N\right) \simeq \prod_{i\in I} \mathrm{Hom}(P_i, N)$$

for any R-module N. To conclude, it just suffices to recall that sequences $0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$ of R-modules are exact if and only if the associated "product sequence"

$$0 \to \prod_{i\in I} A_i \xrightarrow{\prod_{i\in I}\alpha_i} \prod_{i\in I} B_i \xrightarrow{\prod_{i\in I}\beta_i} \prod_{i\in I} C_i \to 0$$

is exact.

Therefore, we deduce that the desired implication is true whenever $P = F$ is a free module. This immediately extends to direct factors of a free module by what we just proved.

2. Prove the *Snake Lemma*: if $0 \to A \to B \to C \to 0$, $0 \to A' \to B' \to C' \to 0$ are short exact sequences of R-modules, and $\alpha\colon A \to A', \beta\colon B \to B', \gamma\colon C \to C'$ are R-linear maps defining a morphism between the two exact sequences (i.e. such that the resulting diagram commutes), then there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \mathrm{coker}\ \alpha \to \mathrm{coker}\ \beta \to \mathrm{coker}\ \gamma \to 0\ .$$

Solution. The Snake Lemma is thoroughly proven in [2], Lemma 1 (and Corollary 2), section 1.5.

3. (a) Let P be a projective module over a ring R. Show that $\mathrm{Tor}_i^R(M, P) = 0$ for every R-module M and every integer $i > 0$.
   (b) i. Show that an R-module M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for any R-module N.
   ii. Show that an R-module M is flat if and only if $\mathrm{Tor}_i^R(M, N) = 0$ for any R-module N and any integer $i > 0$.

Solution.

(a) If P is projective, a projective resolution of P is given by the exact sequence $\cdots 0 \to 0 \to P \xrightarrow{\text{id}} P \to 0$, from which the conclusion trivially follows.

(b) By definition, M is flat if tensoring with M transforms exact sequences into exact sequences. Therefore, it is automatic by the definition of the Tor functor that flatness of M implies $\text{Tor}_i^R(M,N) = 0$ for any R-module N and any integer $i > 0$.

It thus remains to show that M is flat whenever $\text{Tor}_1^R(M,N) = 0$ for any R-module N. Given a short exact sequence of modules $0 \to N' \to N \to N'' \to 0$, we have a long exact sequence in homology

$$\cdots \to \text{Tor}_1(M,N') \to \text{Tor}_1(M,N) \to \text{Tor}_1(M,N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0 \,.$$

Since by assumption $\text{Tor}_1(M,N'') = 0$, the sequence

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact.

4. Let R be a commutative ring, $x \in R$ a nonzerodivisor. Prove that

$$\text{Tor}_1(R/(x), M) \simeq \{m \in M : xm = 0\}$$

(which incidentally explains the name "Tor", since it is connected with torsion elements in this elementary example).

Solution. It is equivalent to determine $\text{Tor}_1(M, R/(x))$ by commutativity of the functor Tor. We may assume the free resolution of $R/(x)$ to be given by

$\cdots \to 0 \to R \xrightarrow{f} R \to R/(x) \to 0$, where f is the map defined by $f(y) = xy$ for all $y \in R$ (injective because x is a nonzerodivisor). By definition of the Tor functor, we have

$$\text{Tor}_1(M, R/(x)) \simeq \frac{\ker\left(M \otimes R \xrightarrow{\text{id}\otimes f} M \otimes R\right)}{\text{Im}(M \otimes 0 \to M \otimes R)} = \ker\left(M \otimes R \xrightarrow{\text{id}\otimes f} M \otimes R\right) .$$

Identifying $M \otimes R$ with M in the canonical way, the map $\text{id} \otimes f$ becomes the assignment $m \to xm$ for all $m \in M$. The proof is concluded.

5. Let R be a local ring with maximal ideal $\mathfrak{m}$, M an R-module. We say that a free resolution

$$F : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \cdots F_0 \to M \to 0$$

of M is *minimal* if each φ_i has image contained in $\mathfrak{m}F_{i-1}$.

If F is a minimal resolution of M as above and $\text{rank} F_i = b_i$ for all $i \in \mathbb{N}$, then prove that

$$\text{Tor}_i^R(R/\mathfrak{m}, M) \simeq (R/\mathfrak{m})^{b_i} \,.$$

The b_i are called the *Betti numbers* of M, by analogy with the corresponding algebraic topology context, in which F is a chain complex.

Solution. By definition, we have

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) = \frac{\ker\left(R/\mathfrak{m} \otimes R^{b_i} \to R/\mathfrak{m} \otimes R^{b_{i-1}}\right)}{\mathrm{Im}(R/\mathfrak{m} \otimes R^{b_{i+1}} \to R/\mathfrak{m} \otimes R^{b_i})} \ . \qquad (1)$$

To get what we want, it suffices to prove that every map

$$R/\mathfrak{m} \otimes R^{b_i} \to R/\mathfrak{m} \otimes R^{b_{i-1}}$$

is the zero map. If this is so, we may indeed deduce from (1) that

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) = R/\mathfrak{m} \otimes R^{b_i} \simeq (R/\mathfrak{m} \otimes R)^{b_i} \simeq (R/\mathfrak{m})^{b_i} \ .$$

The fact that each map $R/\mathfrak{m} \otimes R^{b_i} \to R/\mathfrak{m} \otimes R^{b_{i-1}}$ is the zero map is an immediate consequence of the fact that $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$, since $R/\mathfrak{m}$ is annihilated (as an R-module) by $\mathfrak{m}$.

6. Let R be a noetherian ring, $\mathfrak{m} \subset R$ an ideal, $\hat{\mathfrak{m}}$ the corresponding ideal in the $\mathfrak{m}$-adic completion $\hat{R}$.

   (a) Prove that $\hat{\mathfrak{m}}$ is contained in the Jacobson radical of $\hat{R}$.

   (b) Deduce from the previous point that, if R is a noetherian local ring and $\mathfrak{m}$ is its maximal ideal, then $\hat{R}$ is a (noetherian) local ring with maximal ideal $\hat{\mathfrak{m}}$.

   Solution. For the proof of the two statements, see [1], Proposition 10.15 *iv*) and 10.16.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] S.Bosch (2012), *Algebraic Geometry and Commutative Algebra*, Springer.

[3] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.