Solutions Sheet 12

Injective modules, Ext functor and Artin rings

Let R be a commutative ring, k an algebraically closed field.

1. Let M be a finitely presented module over a ring R, $\varphi\colon R^n \to M$ a surjective R-linear map. Prove that $\ker\varphi$ is finitely generated.

   (Hint: use Snake Lemma, Exercise 2 Sheet 11.)

   Solution. Since M is finitely presented, there is a short exact sequence

   $$0 \to \ker\psi \to R^m \xrightarrow{\psi} M \to 0\ ,$$

   where $\ker\psi$ is a finitely generated submodule of R^m. We also have the short exact sequence given by

   $$0 \to \ker\varphi \to R^n \xrightarrow{\varphi} M \to 0\ .$$

   Let $\gamma\colon M \to M$ be the identity map. Since free modules are projective, there exists an R-linear map $\beta\colon R^m \to R^n$ making the right-hand square of the diagram commute, i.e. such that $\gamma \circ \psi = \varphi \circ \beta$. The restriction of β to $\ker\psi$ gives then a map $\alpha\colon \ker\psi \to \ker\varphi$ making also the left-hand square of the diagram commute. The Snake Lemma gives an isomorphism between the finitely generated R-modules $\ker\varphi/\mathrm{Im}(\alpha) \simeq R^n/\mathrm{Im}(\beta)$. Since $\mathrm{Im}(\alpha)$ is also finitely generated, being an homomorphic image of $\ker\psi$, we conclude that $\ker\varphi$ is finitely generated. See also the proof of Proposition 7, section 1.5 in [2].

2. An R-module Q is called *injective* if, for every monomorphism of R-modules $\alpha\colon N \to M$ and every homomorphism of R-modules $\beta\colon N \to Q$, there exists an homomorphism of R-modules $\gamma\colon M \to Q$ such that $\beta = \gamma \circ \alpha$.

   (a) Prove the following statement: let Q be an R-module, and assume that for every ideal $I \subset R$ and every homomorphism of R-modules $\beta\colon I \to Q$ there is an extension of β to an R-module homomorphism $R \to Q$. Then Q is injective.

   (Hint: use Zorn's lemma to construct the desired extension.)

   (b) Use the previous point to show that an abelian group Q is an injective $\mathbb{Z}$-module if and only if it is *divisible*, i.e. for every $q \in Q$ and every $0 \neq n \in \mathbb{Z}$ there exists $q' \in Q$ such that $nq' = q$.

   Solution. For (a), see [3], Lemma $A3.4$. For (b), see [3], Proposition $A3.5$.

3. Given an R-module M, an *injective resolution* of M is an exact sequence of R-modules

$$0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots$$

in which the $Q_i, i \geqslant 0$ are injective modules.

(a) Assuming (without proving it) that every module can be embedded into an injective module, prove that any R-module admits an injective resolution.

(b) Give an example of an injective resolution of $\mathbb{Z}$ as $\mathbb{Z}$-module.

(Hint: an immediate consequence of point (*b*) *of the previous exercise is that, if Q is an injective abelian group and K is a subgroup, then Q/K is an injective abelian group.)*

Solution.

(a) Let M be an R-module, and embed it into an injective module Q_0. Then, let Q_1 be an injective module in which the cokernel Q_0/M embeds. Continuing in this fashion an injective resolution of M is obtained.

(b) The additive group $\mathbb{Q}$ is an injective $\mathbb{Z}$-module by Exercise 2 (*b*); in particular, the quotient group $\mathbb{Q}/\mathbb{Z}$ is also injetive as a $\mathbb{Z}$-module. Therefore, an injective resolution of $\mathbb{Z}$ is given by

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to 0 \to \cdots .$$

4. The purpose of this exercise is to give another example of a derived functor (we already saw the functor Tor, which is left-derived, in the lecture).

The functor $\operatorname{Hom}_R(M, -)$, where R is a commutative ring and M is a fixed R-module, is left-exact, i.e. it transforms exact sequences of the form $0 \to N' \to N \to N''$ into exact sequences of the same form. If N is an arbitrary R-module, let

$$I : 0 \to N \to I_0 \to I_1 \to \cdots$$

be an injective resolution of N (it always exists by the previous exercise), and form the associated complex

$$\operatorname{Hom}_R(M, I) : \ 0 \to \operatorname{Hom}_R(M, I_0) \to \operatorname{Hom}_R(M, I_1) \to \cdots \quad .$$

We define the *Ext* functor $\operatorname{Ext}^i_R(M, N)$ to be the homology module $H_{-i}(\operatorname{Hom}_R(M, I))$, for all integer $i \geqslant 0$. As in the case of the Tor functor, it can be shown that the definition does not depend on the choice of the injective resolution of N.

(a) Let $x \in R$ be a nonzerodivisor. For any R-module M, compute $\operatorname{Ext}^i_R(R/(x), M)$. In particular, compute $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for any integers n, m.

(b) Prove that a finitely generated abelian group A is free (as a $\mathbb{Z}$-module) if and only if $\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$.

Solution.

(a) As remarked in [3], section $A3.11$, the modules $\mathrm{Ext}^i_R(R/(x), M)$ can be computed also from a projective resolution $F : \cdots \to F_1 \to F_0 \to R/(x) \to 0$ of $R/(x)$; more specifically,

$$\mathrm{Ext}^i_R(R/(x), M) = H_{-i}(\mathrm{Hom}_R(F, M)) ,$$

where $\mathrm{Hom}_R(F, M)$ is the complex

$$\mathrm{Hom}_R(F, M) : 0 \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_1, M) \to \cdots .$$

Now, since x is a nonzerodivisor in R, a projective resolution of $R/(x)$ is given by $\cdots 0 \to 0 \to R \xrightarrow{f} R \to R/(x) \to 0$, where f denotes multiplication by x in R. Applying $\mathrm{Hom}_R(-, M)$ gives the complex

$$0 \to \mathrm{Hom}_R(R/(x), M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(R, M) \to 0 \to 0 \to \cdots ,$$

which identifies canonically with the complex

$$0 \to \{m \in M : xm = 0\} \to M \xrightarrow{g} M \to 0 \to 0 \to \cdots ,$$

where g denotes multiplication by x in the module M. It thus follows immediately that

$$\mathrm{Ext}^i_R(R/(x), M) = \begin{cases} \{m \in M : xm = 0\} & \text{if } i = 0; \\ M/xM & \text{if } i = 1; \\ 0 & \text{if } i > 1. \end{cases}$$

Specializing to the case $R = \mathbb{Z}$, $x = n$, $M = \mathbb{Z}/m\mathbb{Z}$ (viewed as a $\mathbb{Z}$-module), we get

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \begin{cases} \{k \in \{0, 1 \ldots, m-1\} \simeq \mathbb{Z}/m\mathbb{Z} : m|nk\} & \text{if } i = 0; \\ \mathbb{Z}/(m, n)\mathbb{Z} & \text{if } i = 1; \\ 0 & \text{if } i > 1. \end{cases}$$

(b) Assume that A is a finitely generated abelian group which is free as a $\mathbb{Z}$-module. Thus a free resolution of A is given by $\cdots 0 \to 0 \to A \xrightarrow{\mathrm{id}} A \to 0$. Hence $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z})$ is a quotient of $\mathrm{Hom}_{\mathbb{Z}}(0, \mathbb{Z}) = 0$, and is therefore trivial.

Conversely, if a finitely generated abelian group A satisfies $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$, then by the fundamental theorem for finitely generated abelian groups we have that $A \simeq \mathbb{Z}^n \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_r\mathbb{Z}$, where $n \geqslant 0$, $q_i \geqslant 1$, $r \geqslant 0$ are integers. By the analogous property for the Hom functor, it can be easily seen that

$$0 = \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})^n \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q_1\mathbb{Z}, \mathbb{Z}) \oplus \cdots \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q_r\mathbb{Z}, \mathbb{Z}) .$$

An easy computation, similar to the one in the previous point, shows that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$,while $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q_i\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/q_i\mathbb{Z}$. Thus, necessarily, $q_i = 1$ for all $\leqslant i \leqslant r$, so that $A \simeq \mathbb{Z}^n$ is free.

5. Let R be a noetherian ring. Prove that the following are equivalent:

   (a) R is an Artin ring;

   (b) $\mathrm{Spec}(R)$ is discrete (w.r.t. the Zariski topology) and finite;

   (c) $\mathrm{Spec}(R)$ is discrete (w.r.t. the Zariski topology).

   Solution. $[(b) \implies (c)]$ Obvious.

   $[(c) \implies (a)]$ If $\mathrm{Spec}(R)$ is discrete, then in particular every point is closed. Since closed points in the spectrum correspond to maximal ideals, this shows that every prime ideal of R is maximal. Thus $\dim R = 0$, and since R is noetherian by assumption, we deduce that R is an Artin ring.

   $[(a) \implies (b)]$ If R is an Artin ring, then every prime ideal is maximal, so that every point of $\mathrm{Spec}(R)$ is closed for the Zariski topology; moreover R has just a finite number of prime ideals, so that $\mathrm{Spec}(R)$ is a finite set. The unique topology on a finite set for which every singleton is closed is the discrete topology.

6. Let k be a field, and consider the ring $R = k[X^2, X^3]/(X^n)$, where n is a sufficiently large integer (e.g. $n \geqslant 10$). Prove that R has just one prime ideal and conclude that it is zero-dimensional.

   Solution. First of all, let us remark that $k[X^2, X^3]$ is the subring of $k[X]$ consisting of all polynomials with vanishing first-order term. This is because every integer $n \geqslant 2$ can be written as a linear combination $2\alpha + 3\beta$, where $\alpha, \beta \geqslant 0$ are integers, hence any power $X^n, n \geqslant 2$ lies in $k[X^2, X^3]$. To prove the given statement, it is sufficient to show that the maximal ideal (X^2, X^3) (consisting of all polynomials with vanishing zero-order and first-order term) of $k[X^2, X^3]$ is the unique prime ideal containing the principal ideal (X^n), $n \geqslant 2$. Thus, let $\mathfrak{p}'$ be a prime ideal of $k[X^2, X^3]$ containg $X^n = X^2X^{n-2} = X^3X^{n-3}$. If $\mathfrak{p}'$ contains both X^2 and X^3, we are done by maximality of (X^2, X^3), otherwise, by primality of $\mathfrak{p}'$, either $X^{n-2} \in \mathfrak{p}'$ or $X^{n-3} \in \mathfrak{p}'$. The proof is completed by induction on $n \geqslant 2$, the case $n = 2$ being trivial.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] S.Bosch (2012), *Algebraic Geometry and Commutative Algebra*, Springer.

[3] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.