Solutions Sheet 13

Dimension theory, valuation rings and Dedekind domains

1. Let R be a commutative ring. Show that the n-th symbolic power $\mathfrak{p}^{(n)}$ of a prime ideal $\mathfrak{p} \subset R$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$ (here $n \geqslant 1$ is an integer).

   Solution. First, we show that $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal. We know that the extension $\mathfrak{p}^e$ of $\mathfrak{p}$ in $R_\mathfrak{p}$ is a maximal ideal, which implies that its n-th power $(\mathfrak{p}^e)^n$ is primary. Now
   $$\mathfrak{p}^{(n)} = (\mathfrak{p}^n)^{ec} = ((\mathfrak{p}^n)^e)^c = ((\mathfrak{p}^e)^n)^c ,$$
   and since contraction preserves the property of being primary, we conclude that $\mathfrak{p}^{(n)}$ is primary. Let us know compute its radical:
   $$r(\mathfrak{p}^{(n)}) = r((\mathfrak{p}^n)^{ec}) = (r(\mathfrak{p}^n)^e)^c = (r(\mathfrak{p}^n))^{ec} = \mathfrak{p}^{ec} = \mathfrak{p} ,$$
   thus $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.

   If $\mathfrak{q} \supset \mathfrak{p}^n$ is $\mathfrak{p}$-primary and $x \in \mathfrak{p}^{(n)}$, there exists $s \in R \setminus \mathfrak{p}$ such that $sx \in \mathfrak{p}^n \subset \mathfrak{q}$. Since $s \notin r(\mathfrak{q}) = \mathfrak{p}$, we have $s^m \notin \mathfrak{q}$ for all integer $m \geqslant 1$. As $\mathfrak{q}$ is primary, it follows that $x \in \mathfrak{q}$. We have shown that $\mathfrak{p}^{(n)} \subset \mathfrak{q}$, so that $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$.

2. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $k := A/\mathfrak{m}$ denote its residue field. Let $f_1, \dots, f_r \in \mathfrak{m}$. Set $\overline{A} := A/(f_1, \dots, f_r)$. Let $\overline{\mathfrak{m}} \subset \overline{A}$ denote the image of $\mathfrak{m}$.

   (a) Show that $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) \geqslant \dim(\overline{A}) \geqslant d - r$.

   (b) Assume that A is regular. Let $\overline{f_1}, \dots, \overline{f_r} \in \mathfrak{m}/\mathfrak{m}^2$ denote the images of $f_1, \dots, f_r$. Show that the following are equivalent:

      i. $\overline{A}$ is regular of dimension $d - r$.

      ii. $\overline{f_1}, \dots, \overline{f_r}$ are linearly independent over k.

   Solution.

   (a) Since $(A, \mathfrak{m})$ is noetherian and local, $(\bar{A}, \bar{\mathfrak{m}})$ is also noetherian and local. Therefore, Corollary 8 in section 2.4 of [2] gives us that $\dim \bar{A} \leqslant \dim_{\bar{A}/\bar{\mathfrak{m}}} \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$, which gives the first inequality (noticing that $k \simeq \bar{A}/\bar{\mathfrak{m}}$).

   We now want to prove that $\dim \bar{A} \geqslant d - r$. Call s the dimension of $\bar{A}$, and choose $y_1, \dots, y_s \in A$ whose images $\bar{y}_1, \dots, \bar{y}_s \in \bar{A}$ form a system of parameters for $\bar{\mathfrak{m}}$. In particular, $\bar{\mathfrak{m}}$ is the only prime containing $(\bar{y}_1, \dots, \bar{y}_s)$. It follows that $\mathfrak{m}$ is the only prime containing $(f_1, \dots, f_r, y_1, \dots, y_s)$. From this we deduce the upper bound $\dim A \leqslant r + s$.

(b) The proof of Proposition 18 in section 2.4 of [2] shows that ii. is equivalent to the fact that $f_1, \ldots, f_r$ can be extended to a regular system of parameters for A (specifically $f_1, \ldots, f_r$ can be extended to a regular system of parameters for A if and only if $\bar{f}_1, \ldots, \bar{f}_r$ can be extended to a k-basis of $\mathfrak{m}/\mathfrak{m}^2$). Thus, if this condition is met, let $f_1, \ldots, f_r, f_{r+1}, \ldots, f_d$ be a regular system of parameters for A. Then $\dim \bar{A}$ is $d-r$ and the $d-r$ elements $\bar{f}_{r+1}, \ldots, \bar{f}_d$ generate $\bar{\mathfrak{m}}$, so that $\bar{A}$ is regular.

Conversely, assume that $\bar{A}$ is regular of dimension $d-r$, and let $f_{r+1}, \ldots, f_d$ be elements of $\mathfrak{m}$ whose images in $\bar{\mathfrak{m}}$ form a regular system of parameters for $\bar{A}$. If $x \in \mathfrak{m}$, then $x \equiv \sum_{i=r+1}^{d} c_i f_i$ modulo $(f_1, \ldots, f_r)$, for some coefficients $c_i \in A$. In other words, $x - \sum_{i=r+1}^{d} c_i f_i \in (f_1, \ldots, f_r)$. We infer from this that $f_1, \ldots, f_d$ generate $\mathfrak{m}$, whence $f_1, \ldots, f_r$ can be extended to a regular system of parameters for A.

3. Let k be a field, and denote by $R = k[[X_1, \ldots, X_n]]$ the ring of formal power series in n variables with coefficients in k. Prove that R is a regular, noetherian local ring of dimension n.

Solution. Recall that R is the completion of the polynomial ring $k[X_1, \ldots, X_n]$ with respect to the maximal ideal $(X_1, \ldots, X_n)$. Since $k[X_1, \ldots, X_n]$ is noetherian by the Hilbert basis theorem, it follows that R is also noetherian. We also know (from the section on completions) that $\mathfrak{m} = (X_1, \ldots, X_n)$ is the unique maximal ideal of R, which is therefore local. Moreover, it is clear that $\mathfrak{m}/\mathfrak{m}^2 \simeq \bigoplus_{i=1}^{n} k x_i$, i.e. it is an n-dimensional vector space over k. By Corollary 8 in section 2.4 of [2], we have $\dim R \leqslant n$. On the other hand, there is a chain of prime ideals consisting of $(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_n)$, so that $\dim R \geqslant n$. We conclude that $\dim R = n = \dim_k \mathfrak{m}/\mathfrak{m}^2$, which also tells us that R is regular (see Proposition 18, section 2.4 in [2]).

4. Let R be a Dedekind domain, $\mathfrak{a} \neq 0$ an ideal in R.

(a) Show that every ideal in $R/\mathfrak{a}$ is principal.

(b) Deduce that any ideal in R can be generated by at most 2 elements.

Solution.

(a) Assume first that $\mathfrak{p} \neq 0$ is a prime ideal. Then, for any integer $n \geqslant 1$, we have $R/\mathfrak{p}^n \simeq R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}}$, and $R_{\mathfrak{p}}$ is a discrete valuation ring (since R is a Dedekind domain), hence a principal ideal domain. Therefore, every ideal of $R/\mathfrak{p}^n$ is principal.

Now let $\mathfrak{a} \neq 0$ be an arbitrary ideal of R; using Proposition 9.1 of [1], we can choose a prime factorization $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$, where the $\mathfrak{p}_i$ are prime. The canonical map $R \to \prod_i R/\mathfrak{p}_i^{n_i}$ is surjective (because the $\mathfrak{p}_i^{n_i}$ are pairwise coprime in R) with kernel $\mathfrak{a}$. Thus $R/\mathfrak{a}$ is isomorphic to a finite product of principal ideal rings, thus it is a principle ideal ring itself.

(b) Suppose that $\mathfrak{c} \subset R$ is an ideal which is not principal, and let $a \in \mathfrak{c}$ be a given non-zero element. Then the ideal $\mathfrak{c}/(a)$ in $R/(a)$ is principal, generated by some element $b + (a)$ for some $b \in \mathfrak{c}$. It follows that $\mathfrak{c} = (a, b)$.

5. Let G be a totally ordered abelian group, k a field. Denote by R the vector space over k with basis $(e_\alpha)_{0 \leqslant \alpha \in G}$ (i.e, the vector space of formal k-linear combinations of the elements e_α, $0 \leqslant \alpha \in G$). Define a product between basis elements by means of the formula $e_\alpha \cdot e_\beta = e_{\alpha+\beta}$, for every $0 \leqslant \alpha, \beta \in G$, and extend it by k-linearity to the whole of R. Prove that this operation makes R into a valuation ring with value group G and valuation

$$v\left(\sum_\alpha r_\alpha e_\alpha\right) = \min\{\alpha \in G : r_\alpha \neq 0\} ,$$

where the r_α are elements of the field k.

Solution. Let K be the field of fraction of R, and extend v to K by means of the assignment $v(x/y) = v(x) - v(y)$ for any $0 \neq x, y \in R$. It is clear that $v(xy) = v(x) + v(y)$ for any $x, y \neq 0$ in R (simply by the definition of v on R and of the multiplication operation). To conlcude the proof that v is a valuation on K, we need to show the ultrametric inequality $v(x + y) \geqslant \min\{v(x), v(y)\}$, where we may assume without loss of generality that $x, y \in R$ (by the group-morphism property of v). If $x = \sum r_\alpha e_\alpha$, $y = \sum s_\alpha e_\alpha$ and $\alpha_0 \in G$ is the minimal element α of G such that $r_\alpha + s_\alpha \neq 0$, then at least one between r_{α_0} and s_{α_0} is not zero, so that either $v(x) \leqslant \alpha_0$ or $v(y) \leqslant \alpha_0$.

Finally, it is clear by construction that $R = \{x \in K : v(x) \geqslant 0\}$, so that R is a valuation ring with value group G and valuation v.

6. Let R be an integral domain. Prove the following statements:

(a) R is a valuation ring if and only if, for every pair of ideals $\mathfrak{a}, \mathfrak{b} \subset R$, we have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$;

(b) if R is a valuation ring and $\mathfrak{p} \subset R$ is a prime ideal, then $R_\mathfrak{p}$ and $R/\mathfrak{p}$ are both valuation rings.

Solution.

(a) Assume first that R is a valuation ring, and let $\mathfrak{a}, \mathfrak{b} \subset R$ be two ideals. If $\mathfrak{a} \not\subset \mathfrak{b}$, choose an element $f \in \mathfrak{a} \setminus \mathfrak{b}$. For all $g \in \mathfrak{b}$ we know that $f/g \notin R$, otherwise f would be in $\mathfrak{b}$. Since R is a valuation ring, we must have $g/f \in R$, and thus $g \in \mathfrak{a}$. We have shown that $\mathfrak{b} \subset \mathfrak{a}$.

Conversely, if $x = f/g$ is an element of the field of fractions of R, we have that either $(f) \subset (g)$ or $(g) \subset (f)$, so that either $x \in R$ or $x^{-1} \in R$. Hence, R is a valuation ring.

(b) It suffices to apply (a), by the inclusion-preserving correspondence between ideals of $R/\mathfrak{p}$ (resp. $R_{\mathfrak{p}}$) and the ideals of R containing $\mathfrak{p}$ (resp. contained in $\mathfrak{p}$).

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] S.Bosch (2012), *Algebraic Geometry and Commutative Algebra*, Springer.

[3] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.