Solutions Sheet 2

Localization and the Hilbert Nullstellensatz

Let A be a commutative ring, k an algebraically closed field.

1. Let A, B be two commutative rings, $f\colon A \to B$ a ring homomorphism.

   (a) Show that, for every ideal $\mathfrak{b} \subset B$, the preimage $f^{-1}(\mathfrak{b})$ is an ideal in A (it is called the *contraction* of the ideal $\mathfrak{b}$, and it is denoted by $\mathfrak{b}^c$).

   (b) Show that the image $f(\mathfrak{a})$ under f of an ideal $\mathfrak{a} \subset A$ need not be an ideal in B. (For any ideal $\mathfrak{a} \subset A$, the ideal $\mathfrak{a}^e$ of B generated by $f(\mathfrak{a})$ is called the *extension* of the ideal $\mathfrak{a}$.)

   (c) Show that the contraction of a prime ideal is always a prime ideal, while the extension of a prime ideal need not be a prime ideal.

   (d) Prove the following inclusions:

      i. $\mathfrak{a} \subset \mathfrak{a}^{ec}$ for every ideal $\mathfrak{a} \subset A$;

      ii. $\mathfrak{b} \supset \mathfrak{b}^{ce}$ for every ideal $\mathfrak{b} \subset B$;

      iii. $\mathfrak{a}^e = \mathfrak{a}^{ece}$ and $\mathfrak{b}^c = \mathfrak{b}^{cec}$ for any ideals $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$.

   (e) Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset A$, $\mathfrak{b}_1, \mathfrak{b}_2 \subset B$ be ideals. Prove that:

      i. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$, $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c + \mathfrak{b}_2^c$;

      ii. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$, $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$;

      iii. $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$, $(\mathfrak{b}_1\mathfrak{b}_2)^c \supset \mathfrak{b}_1^c\mathfrak{b}_2^c$;

      iv. $r(\mathfrak{a}_1)^e \subset r(\mathfrak{a}_1^e)$, $r(\mathfrak{b}_1)^c = r(\mathfrak{b}_1^c)$.

   Solution.

   (a) Let $\mathfrak{b} \subset B$ be an ideal. Then clearly $f^{-1}(\mathfrak{b})$ is an additive subgroup of A because f is in particular an homomorphism between the additive groups $(A, +)$ and $(B, +)$. Moreover, if $x \in A$ and $a \in f^{-1}(\mathfrak{b})$, then $f(a) \in \mathfrak{b}$, hence $f(xa) = f(x)f(a) \in \mathfrak{b}$, because $\mathfrak{b}$ is an ideal. We deduce that $xa \in f^{-1}(\mathfrak{b})$, which is therefore an ideal.

(b) Take $A = \mathbb{Z}$, $B = \mathbb{Q}$, $f\colon \mathbb{Z} \to \mathbb{Q}$ the canonical inclusion map. Take $\mathfrak{a} = \mathbb{Z}$, the whole ring. We have that $\mathbb{Z}$ is not an ideal in $\mathbb{Q}$, since the latter is a field.

(c) Assume that $\mathfrak{b} \subset B$ is a prime ideal, and let us show that $f^{-1}(\mathfrak{b})$ is prime (we already now that it is an ideal). First, $\mathfrak{b}^{-1}$ is proper since $1 \notin \mathfrak{b}^{-1}$. Then, if $x, y \in A$ are such that $xy \in f^{-1}(\mathfrak{b})$ and $x \notin f^{-1}(\mathfrak{b})$, then $f(xy) = f(x)f(y) \in \mathfrak{b}$ and $f(x) \notin \mathfrak{b}$. Since $\mathfrak{b}$ is prime, this forces $f(y) \in \mathfrak{b}$, which implies $y \in f^{-1}(\mathfrak{b})$. Thus $f^{-1}(\mathfrak{b})$ is prime.

We now want to show that the extension of a prime ideal need not be prime. Again, take $A = \mathbb{Z}$, $B = \mathbb{Q}$, and f the canonical inclusion map. Let $p \in \mathbb{N}$ be a prime number; then the principal ideal $(p) \subset \mathbb{Z}$ generated by p in $\mathbb{Z}$ is prime. However, since $\mathbb{Q}$ is a field, the extension $(p)^e$ must be equal to the whole $\mathbb{Q}$, which is not a proper ideal of itself, hence it cannot be prime.

(d) i. Let $\mathfrak{a} \subset A$ be an ideal. Then

$$\mathfrak{a}^{ec} = f^{-1}(\mathfrak{a}^e) = f^{-1}(\langle f(\mathfrak{a})\rangle) \supset f^{-1}(f(\mathfrak{a})) \supset \mathfrak{a} \ .$$

ii. Let $\mathfrak{b} \subset B$ be an ideal. Then

$$\mathfrak{b}^{ce} = \langle f(\mathfrak{b}^c)\rangle = \langle f(f^{-1}(\mathfrak{b}))\rangle \subset \langle \mathfrak{b}\rangle = \mathfrak{b} \ ,$$

the last equality holding because every ideal clearly coincides with the ideal it generates.

iii. It follows form the previous two points, just as in part (g) of Exercise 14 in the first exercise sheet.

(e) i. First, $\mathfrak{a}_1^e + \mathfrak{a}_2^e$ is an ideal in B containing $f(\mathfrak{a}_1 + \mathfrak{a}_2)$, which by definition of the extension yields $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e + \mathfrak{a}_2^e$. Conversely, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e$ is an ideal containing both $\mathfrak{a}_1^e$ and $\mathfrak{a}_2^e$, hence by definition of the sum we deduce that $\mathfrak{a}_1^e + \mathfrak{a}_2^e \subset (\mathfrak{a}_1 + \mathfrak{a}_2)^e$.

The very same reasoning shows also that $\mathfrak{b}_1^c + \mathfrak{b}_2^c \subset (\mathfrak{b}_1 + \mathfrak{b}_2)^c$.

ii. It is clear that both $\mathfrak{a}_1^e$ and $\mathfrak{a}_2^e$ contain $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e$, which proves that $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$.

Since taking the inverse image preserves all set theoretic operations, we immediately infer from this that $(\mathfrak{b}_1 \cap \mathfrak{b}_1)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$.

iii. The ideal $\mathfrak{a}_1^e\mathfrak{a}_2^e$ contains $f(\mathfrak{a}_1\mathfrak{a}_2)$, hence $(\mathfrak{a}_1\mathfrak{a}_2)^e \subset \mathfrak{a}_1^e\mathfrak{a}_2^e$. Conversely, the ideal $(\mathfrak{a}_1\mathfrak{a}_2)^e$ contains all products xy for $x \in \mathfrak{a}_1^e$ and $y \in \mathfrak{a}_2^e$, hence it must contain the product ideal $\mathfrak{a}_1^e\mathfrak{a}_2^e$.

The last argument applies verbatim to show that $(\mathfrak{b}_1\mathfrak{b}_2)^c \supset \mathfrak{b}_1^c\mathfrak{b}_2^c$.

iv. Let $y = f(x)$ for some $x \in r(\mathfrak{a}_1)$. There exists an integer $n > 0$ such that $x^n \in \mathfrak{a}_1$; thus $y^n = (f(x))^n = f(x^n) \in f(\mathfrak{a}_1) \subset \mathfrak{a}_1^e$, which means that $y \in r(\mathfrak{a}_1^e)$. By definition of the extension, we deduce that $r(\mathfrak{a}_1)^e \subset r(\mathfrak{a}_1^e)$.

Assume $x \in r(\mathfrak{b}_1)^c$, which is equivalent to $f(x)^n = f(x^n) \in \mathfrak{b}_1$ for some integer $n > 0$. Hence $x^n \in f^{-1}(\mathfrak{b}_1) = \mathfrak{b}_1^c$, which gives $x \in r(\mathfrak{b}_1^c)$. The converse inclusion is analogous.

2. In this exercise, we shall examine more closely the particular case of Exercise 1 in which $B = A[S^{-1}]$, the ring of fractions of A with denominators in a multiplicative subset $S \subset A$, and $f\colon A \to A[S^{-1}]$ is the canonical ring homomorphism.

   (a) Let $\mathfrak{a} \subset A$ be an ideal. Show that $\mathfrak{a}^e = \mathfrak{a}[S^{-1}]$, where the latter is defined by

   $$\mathfrak{a}[S^{-1}] = \{as^{-1} : a \in \mathfrak{a},\ s \in S\} \quad .$$

   (b) For any ideal $\mathfrak{b} \subset A[S^{-1}]$, show that there exists an ideal $\mathfrak{a} \subset A$ such that $\mathfrak{a}^e = \mathfrak{b}$. Deduce that if $\mathfrak{b}_1 \subsetneq \mathfrak{b}_2$ are ideals in $A[S^{-1}]$, then $\mathfrak{b}_1^c \subsetneq \mathfrak{b}_2^c$.

   (c) For any ideal $\mathfrak{a} \subset A$, show that equality $\mathfrak{a}^e = A[S^{-1}]$ holds if and only if $\mathfrak{a} \cap S \neq \varnothing$.

   (d) Prove that the map $\mathfrak{p} \mapsto \mathfrak{p}[S^{-1}]$ defines a one-to-one correspondence between prime ideals $\mathfrak{p}$ in A such that $\mathfrak{p} \cap S = \varnothing$ and prime ideals in $A[S^{-1}]$.

   Deduce that, if $\mathcal{R}$ denotes the nilradical ideal of A, then the nilradical ideal of $A[S^{-1}]$ is $\mathcal{R}[S^{-1}]$.

   (e) Assume that $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal. Deduce from the previous point that the prime ideals of the local ring $A_{\mathfrak{p}}$ are in one-to-one correspondence with the prime ideals of A contained in $\mathfrak{p}$.

   Solution.

   (a) The inclusion $\mathfrak{a}[S^{-1}] \subset \mathfrak{a}^e$ is clear. Moreover, $\mathfrak{a}[S^{-1}]$ clearly contains $f(\mathfrak{a})$, so we just need to prove that $\mathfrak{a}[S^{-1}]$ is an ideal in order to get the reverse inclusion. This is immediate from the fact that $\mathfrak{a}$ is an ideal.

   (b) We shall show that $\mathfrak{b}^{ce} = \mathfrak{b}$. We already know from Exercise 1 that $\mathfrak{b}^{ce} \subset \mathfrak{b}$. Let $z = f(x)f(y)^{-1} \in \mathfrak{b}$, with $x \in A$ and $b \in S$. Then clearly $f(x) = f(y)(f(x)f(y)^{-1}) \in \mathfrak{b}$ since $\mathfrak{b}$ is an ideal, thus $x \in \mathfrak{b}^c$ and so $z \in \mathfrak{b}^c[S^{-1}] = \mathfrak{b}^{ce}$ by the previous point.

   In particular, if $\mathfrak{b}_1^c = \mathfrak{b}_2^c$, then $\mathfrak{b}_1 = \mathfrak{b}_1^{ce} = \mathfrak{b}_2^{ce} = \mathfrak{b}_2$.

   (c) We can assume that $\mathfrak{a}$ is a proper ideal of A. Equality $\mathfrak{a}^e = A[S^{-1}]$ holds if and only if every element of $A[S^{-1}]$ can be written as as^{-1} for some $a \in \mathfrak{a}$, $s \in S$. If $\mathfrak{a} \cap S = \varnothing$ then no element $s \in S$ can be put into a form as'^{-1} for some $a \in \mathfrak{a}$ and $s' \in S$, as it can be readily checked. On the other hand, if $a_0 \in \mathfrak{a} \cap S$ then $xs^{-1} = (a_0x)(a_0s)^{-1} \in \mathfrak{a}[S^{-1}]$ for any $x \in A$, $s \in S$.

   (d) We want to show that the contraction map $\mathfrak{p} \mapsto \mathfrak{p}^c$ is an inverse of the given map when restricted to the set of prime ideals of $A[S^{-1}]$. Take a prime ideal $\mathfrak{p}$ in $A[S^{-1}]$. Then the contraction $\mathfrak{p}^c$ is a prime ideal in A by Exercise 1. Since $\mathfrak{p}^{ce} = \mathfrak{p}$ is a proper ideal, by the previous point we have that $\mathfrak{p}^c \cap S = \varnothing$.

It only remains to show that, for any prime ideal $\mathfrak{p} \subset A$ with $\mathfrak{p} \cap S = \varnothing$, $\mathfrak{p}^e$ is a prime ideal in $A[S^{-1}]$ and $\mathfrak{p}^{ec} = \mathfrak{p}$. First, $\mathfrak{p}^e$ is a proper ideal by the previous point. Next, assume $(xs^{-1})(ys'^{-1}) \in \mathfrak{p}[S^{-1}]$; this means that there exist $a \in \mathfrak{p}$, $s'', s''' \in S$ such that $s''s'''xy = ass' \in \mathfrak{p}$. Since $s''s''' \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, we deduce that $xy \in \mathfrak{p}$, which in turn implies that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Hence, either $xs^{-1} \in \mathfrak{p}[S^{-1}]$ or $ys'^{-1} \in \mathfrak{p}[S^{-1}]$. We have thus shown that $\mathfrak{p}[S^{-1}]$ is prime.

Let us now show that $\mathfrak{p}^{ec} \subset \mathfrak{p}$ for all prime ideal $\mathfrak{p} \subset A$ with $\mathfrak{p} \cap S = \varnothing$. Let $a \in \mathfrak{p}^{ec}$, hence $f(a) \in \mathfrak{p}[S^{-1}]$. By definition of the equivalence relation defining $A[S^{-1}]$, we deduce that there exist $a' \in \mathfrak{p}$ and $s, s' \in S$ such that $s'(as - a') = 0$, or equivalently $ss'a = a' \in \mathfrak{p}$. Since $ss' \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, this forces $a \in \mathfrak{p}$, which is what we wanted.

Finally, the last assertion concerning the nilradical should be clear given what we have shown so far and the fact that the nilradical of a ring is the intersection of all its prime ideals.

(e) It is immediate from the previous point.

3. Let A be a commutative ring. Prove that the following two assertions are equivalent:

   (a) A is a noetherian ring.

   (b) A satisfies the *ascending chain condition on ideals*, i.e. for all ascending sequence $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n \subset \cdots$ of ideals of A, there exists an integer $m \geqslant 1$ such that $\mathfrak{a}_m = \mathfrak{a}_{m+j}$ for all integer $j \geqslant 0$.

Use this characterization of noetherian rings and Exercise 2(b) (of the current sheet) to show that if A is noetherian, then so is $A[S^{-1}]$ for any multiplicative subset $S \subset A$.

Solution. (a) $\Rightarrow$ (b). Assume A is noetherian, and let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ an ascending sequence of ideals. Consider $\mathfrak{a}$ to be the union of all ideals $\mathfrak{a}_n$, $n \geqslant 1$. Then $\mathfrak{a}$ is an ideal of A, being an increasing union of ideals, hence it is finitely generated. Assume $\mathfrak{a} = \langle x_1, \ldots, x_n \rangle$ (notation for the ideal generated by $x_1, \ldots, x_n$). Then, for all $1 \leqslant i \leqslant n$, there exists $j_i \geqslant 1$ such that $x_i \in \mathfrak{a}_{j_i}$. Hence $\{x_1, \ldots, x_n\} \subset \mathfrak{a}_m$, where $m = \max_{1 \leqslant i \leqslant n} j_i$. In particular, we deduce that $\mathfrak{a} \subset \mathfrak{a}_m$, whence $\mathfrak{a} = \mathfrak{a}_m = \mathfrak{a}_{m+j}$ for all integer $j \geqslant 0$.

(b) $\Rightarrow$ (a). Assume that A satisfies the ascending chain condition on ideals. Let $\mathfrak{a} \subset A$ be an ideal; we want to show that it has to be finitely generated. Assume for the purpose of a contradiction that it is not. Let $x_1 \in \mathfrak{a}$. Then, there exists $x_2 \in \mathfrak{a} \smallsetminus \langle x_1 \rangle$ (because $\mathfrak{a}$ cannot be generated by x_1). Again, we may pick $x_3 \in \mathfrak{a} \smallsetminus \langle x_1, x_2 \rangle$. Proceeding like this, we are able to construct a sequence $(x_n)_{n \geqslant 1}$ of elements in A such that $x_{n+1} \notin \langle x_1, \ldots, x_n \rangle$. Define $\mathfrak{a}_n = \langle x_1, \ldots, x_n \rangle$ for all $n \geqslant 1$. In this way we obtain an ascending sequence of ideals; therefore there exists $m \geqslant 1$ such that $\mathfrak{a}_m = \mathfrak{a}_{m+j}$ for all $j \geqslant 0$. This means that $x_{m+j} \in \mathfrak{a}_m$ for all $j \geqslant 1$, leading to a contradiction.

Now assume that A is noetherian, and let $S \subset A$ a multiplicatively closed subset of A. On order to prove that $A[S^{-1}]$ is noetherian, we shall make use of the previous characterization and prove that every ascending chain of ideals is eventually constant.

We argue by contradiction: assume that there exists $\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \cdots$, an ascending sequence of ideals in $A[S^{-1}]$ in which every inclusion $\mathfrak{b}_n \subsetneq \mathfrak{b}_{n+1}$ is strict. By Exercise 2(b) of the current sheet, the ascending sequence of ideals $\mathfrak{b}_1^c \subset \mathfrak{b}_2^c \subset \cdots$ in A has the property that each inclusion is strict. However, this contradicts the assumption that A in noetherian.

4. Let $R = A[X]$ be the ring of polynomials in one variable over the commutative ring A. Consider the multiplicative subset $S = \{1, X, X^2, \ldots, X^m, \ldots\}$. The ring of fractions $R[S^{-1}]$ is called the *ring of Laurent polynomials* over A.

   Let $A^{(\mathbb{Z})} = \{f \colon \mathbb{Z} \to A : f(n) = 0 \text{ for all but a finite number of } n\}$, which is a commutative ring when endowed with pointwise addition of functions and multiplication given by

$$(f \cdot g)(n) = \sum_{i+j=n} f(i)g(j) \text{ for all } n \in \mathbb{Z} \text{ , for all } f, g \in A^{(\mathbb{Z})} \text{ .}$$

   Prove that $R[S^{-1}]$ and $A^{(\mathbb{Z})}$ are isomorphic as rings.

   Solution. There is a canonical ring homomorphism $\varphi \colon R \to A^{(\mathbb{Z})}$, defined as follows: for any $f = \sum_{i=0}^n a_i X^i \in R$, set $(\varphi(f))(i) = a_i$ for all $0 \leqslant i \leqslant n$, $(\varphi(f))(i) = 0$ otherwise. Now $\varphi(S) \subset (A^{(\mathbb{Z})})^\times$, i.e. $\varphi(f)$ is invertible in $A^{(\mathbb{Z})}$ for all $f \in S$. Indeed, this is clearly true for $m = 0$. Let $m \in \mathbb{N}_{>0}$, and define the function $g_m \in A^{(\mathbb{Z})}$ by setting $g_m(n) = 1$ if $n = -m$, $g_m(n) = 0$ otherwise. Then it can be easily checked that $\varphi(X^m) \cdot g_m = 1$ in $A^{\mathbb{Z}}$, so that g_m is the inverse of $\varphi(X^m)$.

   By the universal property of the ring of fractions, there exists a ring homomorphism $\overline{\varphi} \colon R[S^{-1}] \to A^{(\mathbb{Z})}$ such that, if $i \colon R \to R[S^{-1}]$ denotes the canonical map, $\overline{\varphi} \circ i = \varphi$. We claim that $\overline{\varphi}$ gives the desired ring isomorphism.

   We first show that $\overline{\varphi}$ is injective: for this, assume $\overline{\varphi}(f/X^m) = 0$ for some $f \in R$, $m \in \mathbb{N}$. Now by definition of the map $\overline{\varphi}$ we have

$$0 = \overline{\varphi}(f/X^m) = \varphi(f)(\varphi(X^m))^{-1} = \varphi(f)g_m \text{ .}$$

   Expliciting pointwise the equality $0 = \varphi(f)g_m$, one readily obtains that $\varphi(f) = 0$. Since φ is clearly injective, we deduce that $f = 0$. Therefore $\overline{\varphi}$ is injective.

   To prove that $\overline{\varphi}$ is surjective, we construct a right inverse for it, i.e. a map $\psi \colon A^{(\mathbb{Z})} \to R[S^{-1}]$ such that $\overline{\varphi} \circ \psi = \mathrm{id}_{A^{(\mathbb{Z})}}$. For all $f \colon \mathbb{Z} \to A$ with finite support, define

$$\psi(f) = \sum_{n\geqslant 0} f(n)X^n + \sum_{n\geqslant 1} f(-n)(X^n)^{-1} ,$$

where both are sums with a finite number of terms since f has finite support. It is straightfoward to verify that $\overline{\varphi}(\psi(f)) = f$ for all function $f \in A^{(\mathbb{Z})}$.

5. Let A be a commutative ring, and assume that for any prime ideal $\mathfrak{p} \subset A$ the localization $A_{\mathfrak{p}}$ at $\mathfrak{p}$ is an integral domain. Is it true then that A is an integral domain?

   Solution. A need not be an integral domain. As a counterexample, take $A = \mathbb{Z}/6\mathbb{Z}$, which has only two non-trivial ideals, namely $\mathfrak{p}_1 = 2\mathbb{Z}/6\mathbb{Z}$ and $\mathfrak{p}_2 = 3\mathbb{Z}/6\mathbb{Z}$. Being evidently maximal, they are both prime ideals.

   It is clear that A is not an integral domain, since for example $\overline{2}, \overline{3} \neq 0$ but $\overline{2} \cdot \overline{3} = \overline{6} = 0$. However, both $A_{\mathfrak{p}_1}$ and $A_{\mathfrak{p}_2}$ are integral domains. We shall prove it only for $A_{\mathfrak{p}_1}$, since the other case is perfectly similar.

   Actually, we claim that $A_{\mathfrak{p}_1}$ is a field (hence, in particular, an integral domain). Recall that $A_{\mathfrak{p}_1}$ is a local ring with unique maximal ideal ${\mathfrak{p}_1}_{\mathfrak{p}_1} = \{xy^{-1} : x \in \mathfrak{p}_1,\ y \notin \mathfrak{p}_1\}$ (for example, as an immediate consequence of Proposition 1.6 *i*) of [1]). Now, in our case, there exists an element $x \in A \setminus \mathfrak{p}_1$ such that

$$x \in \mathrm{Ann}(\mathfrak{p}_1) = \{y \in A : ya = 0 \text{ for all } a \in \mathfrak{p}_1\}$$

   (for example $x = \overline{3}$). This, by the definition of the equivalence relation between fractions in $A_{\mathfrak{p}_1}$, implies that the unique maximal ideal ${\mathfrak{p}_1}_{\mathfrak{p}_1}$ is actually the zero ideal. To conclude, simply notice that any local ring for which the unique maximal ideal is the 0 ideal has to be a field.

6. Let A be a commutative ring. Denote by S_0 the set of all non-zero divisors of A.

   (a) Show that S_0 is a multiplicative subset of A. (The ring $A[S_0^{-1}]$ is called the *total ring of fractions* of A).

   (b) Prove that S_0 is the largest multiplicative subset S of A such that the canonical map $A \to A[S^{-1}]$ is injective.

   (c) Show that, in the ring $A[S_0^{-1}]$, each element is either invertible or a zero-divisor.

   Solution.

   (a) Let $a, b \in S_0$. We want to show that ab is not a zero divisor in A. By contradiction, assume that there exists $c \neq 0$ in A such that $(ab)c = 0$. Then $bc \neq 0$ since b is not a zero-divisor and $a(bc) = 0$, contradicting the fact that a is not a zero-divisor. Moreover, the identity $1 \in A$ is clearly not a zero-divisor. Therefore S_0 is a multiplicative subset of A.

(b) It is equivalent to prove that, given a multiplicative system $S \subset A$, the canonical ring map $\varphi\colon A \to A[S^{-1}]$ is injective if and only if $S \subset S_0$. Indeed, let $x \in A$ be such that $\varphi(x) = 0$. Then, by definition of the equivalence relation defining $A[S^{-1}]$, there exists $y \in S$ such that $0 = y(x \cdot 1 - 1 \cdot 0) = yx$. Now this implies $x = 0$ if and only if $y \in S_0$, which proves the claim.

(c) An arbitrary element of $A[S_0^{-1}]$ is of the form xy^{-1}, with $x \in A$ and $y \in S_0$. There are two cases:

- $x \in S_0$: in this case, $yx^{-1} \in A[S_0]^{-1}$ is clearly an inverse for xy^{-1}.
- $x \notin S_0$: this implies that there exists $x' \in A \setminus \{0\}$ such that $xx' = 0$. The latter equality holds in $A[S_0]^{-1}$ as well (where we identify as usual x with the fraction $x/1$), and $x' \neq 0$ in $A[S_0]^{-1}$ since the canonical map $\varphi\colon A \to A[S_0^{-1}]$ is injective by the previous point. Hence $xy^{-1} \cdot x' = 0$, thus xy^{-1} is a zero-divisor in $A[S_0^{-1}]$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.