D-MATH
Prof. Paul Nelson

Commutative Algebra

HS 2018

Solutions Sheet 3

Spectrum of a ring, modules and primary decomposition

Let R be a commutative ring, k an algebraically closed field.

1. The *spectrum* of a commutative ring R is defined as the set

$$\operatorname{Spec}(R) = \{P \subset R : P \text{ is a prime ideal}\}$$

The purpose of this exercise is to show that $\operatorname{Spec}(R)$ can be equipped with a topology, called the *Zariski topology*, making it into a compact topological space.

Define a subset $X \subset \operatorname{Spec}(R)$ to be *closed* if it is empty or else if there exists an ideal $\mathfrak{a} \subset R$ such that

$$X = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{a} \subset \mathfrak{p}\} .$$

(a) Prove that, if $X_1, X_2 \subset \operatorname{Spec}(R)$ are closed, then so is $X_1 \cup X_2$.

(b) Prove that, if $(X_i)_i$ is a collection of closed subsets of $\operatorname{Spec}(R)$, then so is $\bigcap_{i \in I} X_i$.

(c) Deduce, from the previous two points, that the complements in $\operatorname{Spec}(R)$ of closed subsets are the open sets for a topology on $\operatorname{Spec}(R)$.

(d) Let $(X_i)_{i \in I}$ be a collection of closed subsets of $\operatorname{Spec}(R)$ with the finite intersection property, namely such that $\bigcap_{j \in J} X_j \neq \varnothing$ for any finite subset $J \subset I$. Show that this implies $\bigcap_{i \in I} X_i \neq \varnothing$.

Deduce that $\operatorname{Spec}(R)$, with the topology defined in (c), is a compact topological space.

(e) Which condition should an ideal $\mathfrak{p} \subset R$ satisfy for the singleton $\{\mathfrak{p}\} \subset \operatorname{Spec}(R)$ to be closed?

(f) Show that $\operatorname{Spec}(R)$ is a T_0-space, i.e. for any two distinct points $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(R)$ either there exists a neighborhood of $\mathfrak{p}_1$ not containing $\mathfrak{p}_2$ or there exists a neighborhood of $\mathfrak{p}_2$ not containing $\mathfrak{p}_1$.

(g) Is $\operatorname{Spec}(R)$ always a Hausdorff topological space?

Solution.

(a) For simplicity let us denote, for any ideal $\mathfrak{a} \subset R$, $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{a} \subset \mathfrak{p}\}$, so that a set $X \subset \operatorname{Spec}(R)$ is closed according to our definition if there exists an ideal $\mathfrak{a} \subset R$ such that $X = V(\mathfrak{a})$.

Let now X_1, X_2 be closed subsets of $\mathrm{Spec}(R)$, so that there exist ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset R$ such that $X_1 = V(\mathfrak{a}_1)$ and $X_2 = V(\mathfrak{a}_2)$. We claim that $X_1 \cup X_2 = V(\mathfrak{a}_1\mathfrak{a}_2)$, so that in particular $X_1 \cup X_2$ is closed in $\mathrm{Spec}(R)$.

Assume first that $\mathfrak{p} \in V(\mathfrak{a}_1\mathfrak{a}_2)$, so that $\mathfrak{a}_1\mathfrak{a}_2 \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies that either $\mathfrak{a}_1 \subset \mathfrak{p}$ or $\mathfrak{a}_2 \subset \mathfrak{p}$ (see Exercise 2, Sheet 1). Hence, either $\mathfrak{p} \in X_1$ or $\mathfrak{p} \in X_2$, or equivalently $\mathfrak{p} \in X_1 \cup X_2$.

Conversely, let $\mathfrak{p} \in X_1$ (the case $\mathfrak{p} \in X_2$ is similar). Then $\mathfrak{a}_1 \subset \mathfrak{p}$, hence $\mathfrak{a}_1\mathfrak{a}_2 \subset \mathfrak{a}_1 \subset \mathfrak{p}$, giving $\mathfrak{p} \in V(\mathfrak{a}_1\mathfrak{a}_2)$.

(b) Let $(X_i)_{i\in I}$ be closed subsets of $\mathrm{Spec}(R)$, so that $X_i = V(\mathfrak{a}_i)$ for ideals $\mathfrak{a}_i \subset R$. We claim that $\bigcap_{i\in I} X_i = V(\sum_{i\in I} \mathfrak{a}_i)$, so that again $\bigcap_{i\in I} X_i$ is closed in the spectrum of R.

If $\mathfrak{p} \in X_i$ for all $i \in I$, then by definition $\mathfrak{p} \supset \mathfrak{a}_i$ for all $i \in I$. Since by definition $\sum_{i\in I} \mathfrak{a}_i$ is the smallest ideal containing all the $\mathfrak{a}_i, i \in I$, we deduce that $\mathfrak{p} \supset \sum_{i\in I} \mathfrak{a}_i$, whence $\mathfrak{p} \in V(\sum_{i\in I} \mathfrak{a}_i)$.

The converse inclusion is immediate since $\sum_{i\in I} \mathfrak{a}_i \supset \mathfrak{a}_j$ for all $j \in I$.

(c) It is clear from the previous two points that the set of all complements of closed sets is closed under arbitrary unions and finite intersection, which is a sufficient condition for it to be a topology on the set $\mathrm{Spec}(R)$.

(d) Assume that $(X_i)_{i\in I}$ is a collection of closed subsets of the spectrum satisfying the finite intersection property. Assume $X_i = V(\mathfrak{a}_i)$ for ideals $\mathfrak{a}_i \subset R$. The fact that $\bigcap_{i\in J} X_i = V(\sum_{i\in J} \mathfrak{a}_i) \neq \varnothing$ for a finite subset $J \subset I$ implies that (and it is actually equivalent to) $\sum_{i\in J} \mathfrak{a}_i$ is a proper ideal of R (because any proper ideal is contained in a prime ideal). Since this holds for any finite subset $J \subset I$, it implies that the sum $\sum_{i\in I} \mathfrak{a}_i$ is a proper ideal (simply use the fact that an ideal is proper if and only if it doesn't contain 1). Hence $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} V_i \neq \varnothing$.

This clearly implies that $\mathrm{Spec}(R)$ is a compact topological space, because the condition expressed before on closed sets is equivalent, by taking complements, to the fact that every open cover admits a finite subcover.

(e) A singleton $\{\mathfrak{p}\}$ is closed, by definition, if and only if there exists an ideal $\mathfrak{a} \subset R$ such that $\mathfrak{p}$ is the only ideal containing $\mathfrak{a}$. This clearly can happen if and only if $\mathfrak{p}$ coincides with every maximal ideal that contains it, which is to say if and only if $\mathfrak{p}$ is a maximal ideal itself.

(f) Assume that $\mathfrak{p}_1 \neq \mathfrak{p}_2$ are distinct prime ideals of R. It suffices to show that either there exists a closed subset of $\mathrm{Spec}(R)$ which contains $\mathfrak{p}_1$ but not $\mathfrak{p}_2$ or a closed subset that contains $\mathfrak{p}_2$ but not $\mathfrak{p}_1$. This amounts to prove that either we can find an ideal $\mathfrak{a} \subset R$ contained in $\mathfrak{p}_1$ but not in $\mathfrak{p}_2$ or an ideal $\mathfrak{b} \subset R$ contained in $\mathfrak{p}_2$ but not in $\mathfrak{p}_1$. Since $\mathfrak{p}_1 \neq \mathfrak{p}_2$, we may assume without loss of generality that there exists an element $a \in \mathfrak{p}_1 \smallsetminus \mathfrak{p}_2$. Then the principal ideal $\langle a \rangle$ generated by a is contained in $\mathfrak{p}_1$ but not in $\mathfrak{p}_2$, which is what we wanted.

(g) In general $\operatorname{Spec}(R)$ is not an Hausdorff space for the Zariski topology. For example, consider $R = k[X_1, \dots, X_n]$, where $n \geqslant 2$ and k is an algebraically closed field. Since there are clearly irreducible varieties in k^n which are not points, we may infer that there are prime ideals in R that are not maximal, which in turn means that there are points in the spectrum which, as singletons, are not closed. Hence the space cannot be Hausdorff (actually it is not even T_1 for the very same reason).

2. Let A, B be two commutative rings, $\varphi \colon A \to B$ a ring homomorphism. For any ideal $\mathfrak{b} \subset B$, denote the ideal $\varphi^{-1}(\mathfrak{b})$ by $\varphi^*(\mathfrak{b})$.

   (a) Show that the assignment $\operatorname{Spec}(B) \ni \mathfrak{p} \mapsto \varphi^*(\mathfrak{p})$ gives a well-defined map $\varphi^* \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$.

   (b) Prove that φ^* is continuous, where both $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ are equipped with the Zariski topology.

   (c) Let $\psi \colon B \to C$ be a ring homomorphism. Show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
   (Hence, in the language of categories, the assignment $R \mapsto \operatorname{Spec}(R)$ defines a *contravariant functor* from the category of commutative rings to the category of topological spaces).

   (d) Assume that φ is surjective. Prove that φ^* is an homeomorphism of $\operatorname{Spec}(B)$ onto the closed subset $X_0 = \{\mathfrak{p} \in \operatorname{Spec}(A) : \ker \varphi \subset \mathfrak{p}\}$ of $\operatorname{Spec}(A)$.

   (e) Deduce from the previous point that, for an arbitrary commutative ring A, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\operatorname{nil}(A))$ are naturally homeomorphic, where $\operatorname{nil}(A)$ denotes the nilradical ideal of A.

   (f) Assume that φ is injective. Prove that $\varphi^*(\operatorname{Spec}(B))$ is dense in $\operatorname{Spec}(A)$.
   More precisely, show that $\varphi^*(\operatorname{Spec}(B))$ is dense in $\operatorname{Spec}(A)$ if and only if $\ker \varphi \subset \operatorname{nil}(A)$.

   Solution.

   (a) The map φ^* is well defined since, as we proved in the second exercise sheet (Exercise 1) $\varphi^{-1}(\mathfrak{b})$ is a prime ideal of A whenever $\mathfrak{b}$ is a prime ideal of B.

   (b) It is sufficient to show that the preimage of each closed set of $\operatorname{Spec}(A)$ is closed in $\operatorname{Spec}(B)$. Let $X \subset \operatorname{Spec}(A)$ be closed, so that there exists an ideal $\mathfrak{a} \subset A$ such that $X = V(\mathfrak{a})$. Then

   $$(\varphi^*)^{-1}(X) = \{\mathfrak{p} \subset B \text{ prime ideal} : \mathfrak{a} \subset \varphi^{-1}(\mathfrak{p})\} = \{\mathfrak{p} \subset B \text{ prime ideal} : \varphi(\mathfrak{a})^e \subset \mathfrak{p}\},$$

   in other words, $(\varphi^*)^{-1}(X) = V(\varphi(\mathfrak{a})^e)$, which by definition is closed in $\operatorname{Spec}(B)$.

   (c) It immediately stems from the fact that $(\psi \circ \varphi)^{-1}(\mathfrak{p}) = \varphi^{-1}(\psi^{-1}(\mathfrak{p}))$ for any ideal $\mathfrak{p} \subset C$.

(d) Suppose φ is surjective. In order to prove that $\varphi^*\colon \operatorname{Spec}(B) \to X_0$ is an homeomorphism, it is sufficient to check that:

- $\varphi^*(\mathfrak{p})$ is an element of X_0 for all $\mathfrak{p} \in \operatorname{Spec}(B)$;
- $\varphi^*\colon \operatorname{Spec}(B) \to X_0$ is bijective;
- φ^* is an closed map, i.e. it takes closed subsets of the domain onto closed subsets of the image.

In particular, the last point will also imply that X_0 is closed.

Let $\mathfrak{p}$ be a prime ideal in B; then, since obviously $0 \in \mathfrak{p}$, we get $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \supset \varphi^{-1}(\{0\}) = \ker\varphi$, hence $\varphi^*(\mathfrak{p}) \in X_0$.

Assume $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(B)$ are such that $\varphi^{-1}(\mathfrak{p}_1) = \varphi^{-1}(\mathfrak{p}_2)$. Surjectivity of φ implies that $\mathfrak{p}_1 = \varphi(\varphi^{-1}(\mathfrak{p}_1)) = \varphi(\varphi^{-1}(\mathfrak{p}_2)) = \mathfrak{p}_2$. Thus φ^* is injective. Furthermore, given a prime ideal $\mathfrak{p} \supset \ker\varphi$, then surjecitvity of φ implies that $\varphi(\mathfrak{p})$ is a prime ideal in B. Moreover, since $\mathfrak{p}$ contains the kernel, it holds that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$, which means that $\mathfrak{p}$ is the image under the map φ^* of the prime ideal $\varphi(\mathfrak{p}) \subset B$. We thus proved that φ^* is surjective when it codomain is restricted to X_0.

Finally, assume that $X \subset \operatorname{Spec}(B)$ is closed, hence there is an ideal $\mathfrak{b} \subset B$ such that $X = V(\mathfrak{b})$. Then

$$\varphi^*(X) = \{\varphi^{-1}(\mathfrak{p}) \subset A : \mathfrak{p} \supset \mathfrak{b},\ \mathfrak{p} \text{ prime ideal of } B\} = \{\mathfrak{p}' \subset A \text{ prime ideal} : \mathfrak{p}' \supset \varphi^{-1}(\mathfrak{b})\}$$

where the second equality follows again from the fact that φ is surjective. Hence $\varphi^*(X) = V(\varphi^{-1}(\mathfrak{b}))$ is closed in $\operatorname{Spec}(A)$.

(e) There is a canonical projection map $\pi\colon A \to A/\operatorname{nil}(A)$, which induces a map $\pi^*\colon \operatorname{Spec}(A/\operatorname{nil}(A)) \to \operatorname{Spec}(A)$. Clearly π is surjective, hence by the previous point we deduce that π^* is an homeomorphism of $\operatorname{Spec}(A/\operatorname{nil}(A))$ onto the set

$$\{\mathfrak{p} \in \operatorname{Spec}(A) : \ker\pi = \operatorname{nil}(A) \subset \mathfrak{p}\}\ . \tag{1}$$

Clearly, any prime ideal contains the nilradical ideal by definition of the latter, therefore the set in (1) is simply $\operatorname{Spec}(A)$. This achieves the proof.

(f) We are going to prove directly the second assertion, which implies as a particular case the first one. By the previous point, we may assume without loss of generality that $\operatorname{nil}(B) = \{0\}$ (if not, replace B with $B/\operatorname{nil}(B)$).

The set $\varphi^*(\operatorname{Spec}(B))$ is dense in $\operatorname{Spec}(A)$ if and only if, for any proper ideal $\mathfrak{a} \subset A$ not contained in $\operatorname{nil}(A)$, there exists a prime ideal $\mathfrak{p} \subset B$ such that $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ does not contain $\mathfrak{a}$ (a set is dense in a topological space if and only if it intersects non-trivially any non-empty open set). Assume that $\ker\varphi \subset \operatorname{nil}(A)$, and fix a non-trivial, proper ideal $\mathfrak{a} \subset A$. It suffices to show that there exists a prime ideal $\mathfrak{p} \subset B$ such that $\varphi(\mathfrak{a}) \not\subseteq \mathfrak{p}$. Assume, for the sake of contradiction, that $\varphi(\mathfrak{a})$ is contained in any prime ideal of B, which means that $\varphi(\mathfrak{a}) \subset \mathrm{nil}(B) = \{0\}$. Hence $\mathfrak{a} \subset \ker\varphi \subset \mathrm{nil}(A)$, contradicting our assumption on $\mathfrak{a}$.

Conversely, assume that $\varphi^*(\mathrm{Spec}(B))$ is dense in $\mathrm{Spec}(A)$. Let $x \in \ker\varphi$, and denote by $\mathfrak{a} = \langle x \rangle$, the principal ideal generated by x. Then it is clear that $\varphi^*(\mathfrak{p}) \supset \mathfrak{a}$ for any $\mathfrak{p} \in \mathrm{Spec}(B)$, which means that $\varphi^*(\mathrm{Spec}(B)) \subset V(\mathfrak{a})$, where the latter is a closed subset of the spectrum of A. Therefore,

$$\mathrm{Spec}(A) = \overline{\varphi^*(\mathrm{Spec}(B))} \subset V(\mathfrak{a}) \subset \mathrm{Spec}(A) ,$$

hence equality holds everywhere. To conclude, simply notice that $V(\mathfrak{a}) = \mathrm{Spec}(A)$ necessarily implies $\mathfrak{a} \subset \mathrm{nil}(A)$, which in turn gives $x \in \mathrm{nil}(A)$.

3. In this exercise we examine a connection between the Zariski topology on the spectrum of a ring and the Zariski topology on the affine space k^n (k algebraically closed field). Let $X \subset k^n$ be a variety, and denote by $I(X)$ the ideal of $k[X_1, \ldots, X_n]$ defined by it. The quotient ring

$$P(X) = k[X_1, \ldots, X_n]/I$$

is called the *(affine) coordinate ring* of X.

   (a) Define $\tilde{P}(X)$ to be the *ring of polynomial functions* on X, namely

   $$\tilde{P}(X) = \{\varphi\colon X \to k : \exists\, f \in k[X_1, \ldots, X_n] \text{ s.t. } \varphi(x) = f(x)\ \forall\, x \in X\} ,$$

   with the obvious addition and multiplication operations. Show that $P(X)$ and $\tilde{P}(X)$ are isomorphic rings.

   (b) For each $x \in X$, denote by $\mathfrak{m}_x$ the ideal of all $f \in P(X)$ such that $f(x) = 0$. Show that it is a maximal ideal in $P(X)$.

   Hint: in the one-to-one correspondence between affine varieties in k^n and radical ideals of $k[X_1, \ldots, X_n]$, prove that, for any variety $X \subset k^n$, $I(X)$ is a maximal ideal whenever $X = \{P\}$ is a singleton.

   (c) Given an arbitrary commutative ring R, we define

   $$\mathrm{Max}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \text{ is a maximal ideal}\} ;$$

   $\mathrm{Max}(R)$ is called the *maximal spectrum* of R.

   In the previous point, we have thus defined a map $\mu\colon X \to \mathrm{Max}(P(X))$. Prove that μ is injective.

   (d) Using the weak form of the Hilbert Nullstellensatz, prove that the map μ is surjective.

   (e) Suppose now that $X = k^n$, so that $P(X) \simeq k[X_1, \ldots, X_n]$. Show that the map μ is continuous with respect to the Zariski topologies on k^n and on $\mathrm{Max}(k[X_1, \ldots, X_n])$. Is μ an homeomorphism onto its image?

Solution.

(a) We define a map $\Phi\colon P(X) \to \tilde{P}(X)$, by sending each equivalence class $[f]$ of a polynomial $f \in k[X_1, \dots, X_n]$ to the polynomial function on X defined by f. The map is well defined, since if $[f] = [g]$ for some $f, g \in k[X_1, \dots, X_n]$, then $f - g \in I(X)$, whence $(f-g)/(x) = 0$ for all $x \in X$. Thus $f(x) = g(x)$ for all $x \in X$.

The map Φ is clearly a ring homomorphism. By definition of the ring $\tilde{P}(X)$, Φ is surjective. It only remains to prove that it is injective. Assume $\Phi([f]) = 0$ for some $f \in k[X_1, \dots, X_n]$. Then $f(x) = 0$ for all $x \in X$, which implies $f \in I(X)$, or equivalently $[f] = 0$.

(b) It suffices to show that the ideal $I(\{x\})$ is maximal in $k[X_1, \dots, X_n]$. This follows from the fact that $\{x\}$ is a minimal variety (in the sense of inclusion) in k^n and the correspondence $X \mapsto I(X)$ is an anti-isomorphism of partially ordered sets.

(c) Assume that $x_1, x_2 \in X$ are such that $\mathfrak{m}_{x_1} = \mathfrak{m}_{x_2}$. Then it is immediate to deduce that $I(\{x_1\}) = I(\{x_2\})$, hence $x_1 = Z(I(\{x_1\})) = Z(I(\{x_2\})) = x_2$. Therefore the map μ is injective.

(d) Let $\mathfrak{m}$ be a maximal ideal in $P(X)$, and denote by $\mathfrak{m}' \subset k[X_1, \dots, X_n]$ its inverse image under the projection map $k[X_1, \dots, X_n] \to P(X)$. Then $\mathfrak{m}'$ is a maximal ideal in $k[X_1, \dots, X_n]$, in particular it is a proper ideal. By the weak form of Hilbert Nullstellensatz, the zero locus $Z(\mathfrak{m}')$ is non-empty. Let $x \in Z(\mathfrak{m}')$. Since $\mathfrak{m}' \supset I(X)$, we deduce that $\{x\} \subset Z(\mathfrak{m}') \subset Z(I(X)) = X$, hence $x \in X$. It is now trivial to check that $\mu(x) = \mathfrak{m}$ for such a choice of x.

(e) Let $C \subset \mathrm{Max}(k[X_1, \dots, X_n])$ be a closed set (for the induced Zariski topology on the maximal spectrum). Then there exists an ideal $I \subset k[X_1, \dots, X_n]$ such that $C = \{\mathfrak{m} \text{ maximal ideal} : I \subset \mathfrak{m}\}$. We claim that $\mu^{-1}(C) = Z(I)$, so that in particular $\mu^{-1}(C)$ is closed in k^n for the Zariski topology and the map μ is continuous.

Assume that $x \in Z(I)$, then $\mu(x) = \mathcal{I}(\{x\}) \supset \mathcal{I}(Z(I)) \supset I$, hence $\mu(x) \in C$. Conversely, assume $\mu(x) \in C$, i.e. $\mu(x) \supset I$. By definition of the map μ, this means that, for all $f \in I$, $f(x) = 0$; thus $x \in Z(I)$.

The map μ is actually an homeomorphism. It suffices to show that the image of each closed set is closed. Let $Z(I)$ be a closed set in k^n, where $I \subset k[X_1, \dots, X_n]$ is an ideal. Then we claim that

$$\mu(Z(I)) = \{\mathfrak{m} \subset k[X_1, \dots, X_n] \text{ maximal ideal} : \mathfrak{m} \supset I\} ,$$

so that it is indeed closed in $\mathrm{Max}(k[X_1, \dots, X_n])$ for the Zariski topology. If $x \in Z(I)$, then $\mu(x) = \mathcal{I}(x) \supset \mathcal{I}(Z(I)) \supset I$. Conversely, if $\mathfrak{m}$ is a maximal ideal containing I, then $\{x\} = Z(\mathfrak{m}) \subset Z(I)$, so that $x \in Z(I)$ and $\mu(x) = \mathfrak{m}$.

4. Let $(M_i)_{i\in I}$ be a collection of modules over the commutative ring R. Denote by $\bigoplus_{i\in I} M_i$ their direct sum, and by $\prod_{i\in I} M_i$ their product. For any $j \in I$, denote by $\pi_j\colon \prod_{i\in I} M_i \to M_j$ the canonical (linear) projection onto the j-th factor, and by $\eta_j\colon M_j \to \bigoplus_{i\in I} M_i$ the injective linear map defined by

$$(\eta_j(x))_i = \begin{cases} x & \text{if } i = j \ ; \\ 0 & \text{if } i \neq j \ . \end{cases}$$

   (a) Prove the *universal property of the product*: for any R-module N and any collection $(\varphi_i)_{i\in I}$ of linear maps $\varphi_i\colon N \to M_i$, there exists a unique linear map $\varphi\colon N \to \prod_{i\in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for all $j \in I$.

   (b) Prove the *universal property of the direct sum*: for any R-module P and any collection $(\psi_i)_{i\in I}$ of linear maps $\psi_i\colon M_i \to P$, there exists a unique linear map $\psi\colon \bigoplus_{i\in I} M_i \to P$ such that $\psi \circ \eta_j = \psi_j$ for all $j \in I$.

   (c) Let $(N_j)_{j\in J}$ be another collection of R-modules. Use the previous two points to prove that there exists a canonical linear isomorphism

$$\operatorname{Hom}\left(\bigoplus_{i\in I} M_i, \prod_{j\in J} N_j\right) \to \prod_{(i,j)\in I\times J} \operatorname{Hom}(M_i, N_j) \ .$$

   Solution. This exercise has been solved in the lecture.

5. In this exercise we discuss the notion of *direct limits of modules.*

   Let $(I, \leqslant)$ be a directed set, i.e. a partially ordered set with the property that for all $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Let $(M_i)_{i\in I}$ be a collection of R-modules, and assume that we are given, for each $i \leqslant j \in I$, an R-module morphism $\mu_{ij}\colon M_i \to M_j$ such that:

   - $\mu_{ii}\colon M_i \to M_i$ is the identity map for all $i \in I$;
   - $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ for all $i \leqslant j \leqslant k \in I$.

   The modules $(M_i)_{i\in I}$ together with the collection $(\mu_{ij})_{i\leqslant j\in I}$ form a so-called *direct system* of modules.

   Denote by C the direct sum of all modules $M_i, i \in I$ and identify each factor M_i with its isomorphic image in C. Let D be the submodule of C generated by the set $\{x_i - \mu_{ij}(x_i) : x_i \in M_i, i \leqslant j \in I\}$. Let $M = C/D$, and let $\mu\colon C \to M$ be the canonical projection map. Denote by μ_i the restriction of μ to M_i The module M is called the *direct limit* of the direct system, and it is denoted by $M = \varinjlim M_i$.

   (a) Prove that $\mu_i = \mu_j \circ \mu_{ij}$ for all $i \leqslant j \in I$.

   (b) Show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

(c) Show that, if $\mu_i(x_i) = 0$ for some $x_i \in M_i$ and some $i \in I$, then there exists $j \geqslant i$ such that $\mu_{ij}(x_i) = 0$.

(d) Prove the *universal property of the direct limit*: for any R-module N and any collection $(\varphi_i)_{i\in I}$ of R-linear maps $\varphi_i \colon M_i \to N$ such that $\varphi_i = \varphi_j \circ \mu_{ij}$ for all $i \leqslant j \in I$, there exists a unique R-linear map $\varphi\colon M \to N$ such that $\varphi_i = \varphi \circ \mu_i$ for all $i \in I$.

Solution.

(a) Let $i \leqslant j \in I$ and $x \in M_i$; we want to show that $\mu_i(x) = \mu_j(\mu_{ij}(x))$. This is clear since x and $\mu_{ij}(x)$ are identified in M, since their difference belongs to D.

(b) Let $[x]$ be an element of M, where $x \in C$, so that $x = \sum_{i\in I} x_i$ for some $x_i \in M_i$ with $x_i = 0$ for all but finitely many indices $i \in I$. Let $J = \{i \in I : x_i \neq 0\}$; then since I is a directed set there exists $j \in I$ such that $j \geqslant i$ for all $i \in J$. Thus, $[x] = [\sum_{i\in J} \mu_{ij}(x_i)]$, since $x - \sum_{i\in J} \mu_{ij}(x_i) \in D$. Now $\sum_{i\in J} \mu_{ij}(x_i) \in M_j$, hence we get what we wanted.

(c) Assume that $\mu_i(x_i) = [x_i] = 0$ for some $x_i \in M_i$, so that $x_i \in D$. We can thus write $x_i = \sum_k \alpha_k(x_{i_k} - \mu_{i_k j_k}(x_{i_k}))$ for some $\alpha_k \in R$, $i_k \leqslant j_k \in I$. We may assume that $\alpha_k = 1$ and that each pair $(i,j) \in I^2$ with $i \leqslant j$ appears only once in the previous expression, since the genrators $x_l - \mu_{lm}(x_l), l \leqslant m \in I$ are clearly closed under addition and multiplication by scalars. We may thus write

$$x_i = \sum_{l\leqslant m} x_l - \mu_{lm}(x_l) = \sum_l \sum_{l\leqslant m} x_l - \mu_{lm}(x_l) \ .$$

Let $A \subset I$ be the finite subset of indexes appearing in the sum of the right-hand side of the previous equality. We may thus write

$$\sum_l \sum_{l\leqslant m} x_l - \mu_{lm}(x_l) = \sum_{s\in A} y_s$$

for some $y_s \in M_s$. Equality $x_i = \sum_{a\in A} y_s$ forces $y_i = x_i$ and $y_s = 0$ for all $s \neq i$. Take $j \in I$ such that $j \geqslant m$ for every m such that the couple $(l,m), l \leqslant m$ belongs to A^2 for some $l \in A$. Then

$$\mu_{ij}(x_i) = \sum_l \sum_{l\leqslant m} \mu_{lj}(x_l) - \mu_{mj}(\mu_{lm}(x_l)) = 0$$

because of the equality $\mu_{mj} \circ \mu_{lm} = \mu_{lj}$.

(d) By the universal property of the direct sum, there exists a linear map $\tilde{\varphi}\colon C \to N$ such that $\tilde{\varphi} \circ \eta_i = \varphi_i$ for all $i \in I$, where $\eta_i\colon M_i \to C$ is the canonical inclusion. The condition $\varphi_i = \varphi_j \circ \mu_{ij}$ for all $i \leqslant j \in I$ precisely means that $D \subset \ker \tilde{\varphi}$, hence $\tilde{\varphi}$ factors through a map $\varphi\colon M \to N$ such that $\varphi \circ \pi = \tilde{\varphi}$, where $\pi\colon C \to M$ is the canonical projection. It is immediate to check that, with this definition, $\varphi_i = \varphi \circ \mu_i$ for all $i \in I$. We have thus shown existence. To show uniqueness, we resort to point (b) of the current exercise. Every element of M is of the form $\mu_i(x_i)$ for some $x_i \in M_i$, hence φ must send this element to $\varphi_i(x_i)$ (because of the property $\varphi_i = \varphi \circ \mu_i$ that φ has to satisfy). The proof is concluded.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.