Solutions Sheet 5

Flatness and primary decomposition

Let R be a commutative ring, k an algebraically closed field.

1. Let R be a ring, $\mathfrak{a} \subset R$ an ideal, M an R-module. Denote by $\mathfrak{a}M$ the submodule of M generated by the elements λx, where $\lambda \in \mathfrak{a}$, $x \in M$. Show that the R-modules $(R/\mathfrak{a}) \otimes_R M$ and $M/\mathfrak{a}M$ are isomorphic.

   (Hint: tensor the exact sequence $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ *with* M*.)*

   Solution. Given the exact sequence
   $$\mathfrak{a} \to R \to R/\mathfrak{a} \to 0 \; ,$$
   we know from the lecture the tensoring with M preserves the exactness property, i.e.
   $$\mathfrak{a} \otimes_R M \to R \otimes_R M \to (R/\mathfrak{a}) \otimes_R M \to 0 \; .$$
   We deduce that $(R/\mathfrak{a}) \otimes_R M$ is isomorphic as an R-module to the quotient module $(R \otimes_R M)/(\mathfrak{a} \otimes_R M)$. The R-linear map $\varphi \colon M \to (R \otimes_R M)/(\mathfrak{a} \otimes_R M)$ sending m to the equivalence class of $1 \otimes m$ is surjective, being the composition of two surjective maps. Moreover, it is immediate to verify that $\ker \varphi = \mathfrak{a}M$, so that we get an isomorphism $(R \otimes_R M)/(\mathfrak{a} \otimes_R M) \simeq M/\mathfrak{a}M$. Combining the two isomorphisms, we get the desired result.

2. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be an exact sequence of R-modules. Assume that M and P are finitely generated. Show that N is finitely generated.

   Solution. Let $S \subset M$ and $T \subset P$ be finite generating sets of M and P respectively. Let $T' \subset N$ be a finite set such that $g(T') = T$. We claim that N is generated by the set $f(S) \cup T'$ (so that, in particular, it is finitely generated). Denote by $N' \subset N$ the submodule of N generated by $f(S) \cup T'$. Then N' contains $f(M) = \ker g$, since S generates M. Moreover, $g(N')$ is a submodule of P containing $g(T') = T$; T being a generating set for P, this implies that $g(N') = P$.

   Thus, given an arbitrary $x \in N$, there exists $y \in N'$ such that $g(x) = g(y)$ (since $g(N') = P$); hence $x - y \in \ker g \subset N'$. We conclude that $x = (x - y) + y \in N'$, as N' is a submodule. We have thus established that $N' = N$, which was our claim.

3. Let R, S be two rings, M an R-module, P an S-module, N an (R, S)-*bimodule*, namely N is simultaneously an R-module and an S-module and the two module structures are compatible, i.e. $(rx)s = r(xs)$ for all $r \in R, s \in S, x \in N$.

(a) Show that $M \otimes_R N$ has a canonical S-module structure, and that $N \otimes_S P$ has a canonical R-module structure.

(b) Prove that, with the structures defined in the previous point, there is an isomorphism of abelian groups

$$(M \otimes_R N) \otimes_S P \simeq M \otimes_R (N \otimes_S P) .$$

(c) Use the previous point to show that, if $f\colon R \to S$ is a ring morphism and M is a flat R-module, then the extended module $M_S = S \otimes_R M$ is a flat S-module.

Solution.

(a) The only reasonable way to define a scalar multiplication on $M \otimes_R N$ by elements of S is by defining $s \cdot (f \otimes g) = f \otimes gs$ for all $s \in S$, $f \in M$ and $g \in N$, and then extending it by linearity on all elements of $M \otimes_R N$. the fact that the two module structures on N are compatible implies that all the axioms of an S-module are satisfied for this scalar multiplication on $M \otimes_R N$. The same applies *mutatis mutandis* to $N \otimes_S P$.

(b) Fix an element $p_0 \in P$, and define the map $f_{p_0}\colon M \times N \to M \otimes_R (N \otimes_S P)$ by $f_{p_0}(m, n) = m \otimes (n \otimes p_0)$. This map is clearly R-bilinear, hence by the universal property of the tensor product it factors through an R-linear map $\tilde{f}_{p_0}\colon M \otimes_R N \to M \otimes_R (N \otimes_S P)$, which thus has the property that $\tilde{f}_{p_0} m \otimes n = m \otimes (n \otimes p_0)$. Now define a map $f\colon (M \otimes_R N) \times P \to M \otimes_R (N \otimes_S P)$ by setting $f(a, p) = \tilde{f}_p(a)$ for all $a \in M \otimes_R N$ and all $p \in P$. The map f is bilinear over the ring S, where $M \otimes_R (N \otimes_S P)$ can be endowed with the structure of an S-module as in the previous map since it is easy to check that $N \otimes_S P$ is an (R, S)-bimodule with the obvious scalar operations. Therefore, we have an induced S-linear map (again, applying the universal property of the tensor product) $\tilde{f}\colon (M \otimes N) \otimes_S P \to M \otimes_R (N \otimes_S P)$ sending $(m \otimes n) \otimes p$ to $m \otimes (n \otimes p)$ for all $m \in M, n \in N$ and $p \in P$. It is not hard to check that the map $\tilde{f}$ is also R-linear.

Applying the very same reasoning as above, but this time starting from $M \otimes_R (N \otimes_S P)$ and constructing a map with values in $(M \otimes_R N) \otimes_S P$, we get an (R, S)-linear map $\tilde{g}\colon M \otimes_R (N \otimes_S P) \to (M \otimes_R N) \otimes_S P$ with the property that the composed maps $\tilde{g} \circ \tilde{f}$ and $\tilde{f} \circ \tilde{g}$ are the identity map on a set of generators of the respective domains. By linearity, they coincide with the identity map everywhere, so that $\tilde{g}$ and $\tilde{f}$ are inverses of each other. We have thus shown that $(M \otimes_R N) \otimes_S P$ and $M \otimes_R (N \otimes_S P)$ are isomorphic not only as abelian groups, but also as (R, S)-bimodules.

(c) It is immediate using the isomorphism of the previous point and the fact that S is flat as an S-module: given any exact sequence of S-modules, first tensorize it over S with S, so that you get again an exact sequence of S-modules, then tensorize the latter over R with M and use the isomorphism of the previous point to get again an exact sequence.

4. (a) Assume that M, N are flat R-modules. Show that $M \otimes_R N$ is a flat R-module.
   (b) If S is a flat R-algebra and N is a flat S-module, then N is flat as an R-module.

   Solution.

   (a) It is an immediate consequence of the associativity of the tensor product.
   (b) Given an exact sequence of R-modules, first tensorize it over R with the flat R-algebra S, so as to obtain again an exact sequence of R-modules. Clearly, the latter is also an exact sequence of S-modules, hence we may tensorize it over S with the flat S-module N to obtain again an exact sequence of S-modules. Use the associativity of the tensor product as stated in the previous exercise and the fact that $S \otimes_S E \simeq E$ for any S-module E to get the result.

5. Let M, N be finitely generated modules over noetherian ring R. The aim here is to prove that
$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N)\ , \tag{1}$$
where $\mathrm{Supp}(M)$ is the set of all primes of R containing the annihilator of M.

   (a) Show that it is sufficient to assume that R is a local ring, i.e. it contains a unique maximal ideal.
   (b) Prove that, if M is a non-zero finitely generated module over a noetherian local ring R with maximal ideal $\mathfrak{p}$, then $\mathrm{Hom}_R(M, R/\mathfrak{p})$ is non-zero.

   (*You may use Nakayama's lemma, without proving it: let $\mathfrak{p}$ be the maximal ideal of a noetherian local ring R, and let M be a finitely generated R-module. If $\mathfrak{p}M = M$, then $M = 0$.*

   Recall also that $M/\mathfrak{p}M$ has a vector space structure over the field $R/\mathfrak{p}$.)
   (c) Conclude.

   Solution.

   (a) Since associated primes behave well under localization (theorem 3.1 *c*) of [1]), it is sufficent to prove (localizing at prime ideals) that, if R is a local ring with maximal ideal $\mathfrak{p}$, then $\mathfrak{p}$ is in $\mathrm{Ass}(\mathrm{Hom}_R(M, N))$ if and only if it is in $\mathrm{Supp}(M) \cap \mathrm{Ass}(N)$.
   (b) Using Nakayama's lemma, we deduce that $\mathfrak{p}M$ is a proper submodule of M, since M is non-zero. By the result proven in Exercise 1 of the current sheet, we have that the non-zero quotient module $M/\mathfrak{p}M$ is isomorphic, as

an R-module, to the tensor product $(R/\mathfrak{p}) \otimes_R M$, which has a canonical $(R/\mathfrak{p})$-module structure, or in other words $M/\mathfrak{p}M$ is a non-zero vector space over the field $k = R/\mathfrak{p}$. This implies that there exists a non-zero k-linear form $\varphi\colon M/\mathfrak{p}M \to k$. Then the map $\varphi \circ \pi$, where $\pi\colon M \to M/\mathfrak{p}M$ is the canonical projection, is a non-zero R-linear map from M to $R/\mathfrak{p}$. The proof is concluded.

(c) Assume first that $\mathfrak{p} \in \mathrm{Supp}(M)$; the previous point gives us the existence of a surjective R-linear map $\psi\colon M \to R/\mathfrak{p}$. If $\mathfrak{p}$ is also in $\mathrm{Ass}(N)$, there there is an injection $\psi'\colon R/\mathfrak{p} \to N$. The composition $f = \psi' \circ \psi$ is a (non-zero) map in $\mathrm{Hom}_R(M, N)$, and it is immediate to check that it is annihilated by $\mathfrak{p}$.

Conversely, if $\mathfrak{p}$ is the annihilator of a non-zero R-linear map $f\colon M \to N$, then surely M is not the zero module, hence $\mathrm{Ann}(M)$ is a proper ideal and thus it is contained in the unique maximal ideal $\mathfrak{p}$, so that $\mathfrak{p} \in \mathrm{Supp}(M)$. It is also trivial to deduce that $\mathfrak{p}$ annihilates $\mathrm{Im}(f)$ from the fact that it annihilates f. This implies that $\mathfrak{p} \in \mathrm{Ass}(N)$.

6. Let R be a commutative ring, J an arbitrary set of indexes. The *ring of polynomials in the set of variables $\{X_j\}_{j\in J}$ with coefficients in* R, denoted by $R[\{X_j\}_{j\in J}]$, is defined as the free R-module over the set $N^{(J)} = \{\alpha\colon J \to \mathbb{N} : \alpha(j) = 0 \text{ for all but finitely many } j \in J\}$. For fixed $\alpha \in \mathbb{N}^{(J)}$, function $f\colon \mathbb{N}^{(J)} \to R$ taking value 1 at α and 0 elsewhere will be denoted by the monomial notation $X_{j_1}^{n_1} \cdots X_{j_m}^{n_m}$, where $n_i \in \mathbb{N}$ and $\{j_1, \dots, j_m\}$ is the support of the function α. Hence $R[\{X_j\}_{j\in J}]$ is simply the set of all finite R-linear combinations of such monomials, with the obvious operations of sum and product (making it into an R-algebra).

The purpose of this exercise is to show that, for modules over an arbitrary (i.e. not necessarily noetherian) ring, the set of associated primes may be empty even if the module is non-zero.

Consider the quotient ring $A = k[\{X_n\}_{n\in\mathbb{N}}]/(x_i^2, i \in \mathbb{N})$ (k arbitrary field).

(a) Show that A has a unique prime ideal, namely the ideal

$$\mathfrak{p} = (x_i, i \in \mathbb{N})/(x_i^2, i \in \mathbb{N}) .$$

(b) Prove that $\mathfrak{p}$ is not associated to the A-module A.

Solution.

(a) It is sufficient show that the unique prime ideal of $k[\{X_n\}_{n\in\mathbb{N}}]$ containing $\mathfrak{a} = (x_i^2, i \in \mathbb{N})$ is $\mathfrak{p}' = (x_i, i \in \mathbb{N})$. Let thus $\mathfrak{b}$ be a prime ideal of $k[\{X_n\}_{n\in\mathbb{N}}]$ containing $\mathfrak{a}$; then, for all $i \in \mathbb{N}$, $x_i^2 \in \mathfrak{b}$ together with primality implies $x_i \in \mathfrak{b}$, whence $\mathfrak{b} \supset \mathfrak{p}'$. On the other hand, it is straightforward to see that the quotient ring $k[\{X_n\}_{n\in\mathbb{N}}]/\mathfrak{p}'$ is isomorphic to the field k, whence $\mathfrak{p}'$ is a maximal ideal. This forces $\mathfrak{b} = \mathfrak{p}'$ and yields the assertion.

(b) Assume by contradiction that $\mathfrak{p}$ annihilates an element $f+(x_i^2, i\in\mathbb{N})$, with $0\neq f\in k[\{X_n\}_{n\in\ltimes}]$. In particular, for any finite subset $L\subset\mathbb{N}$, the polynomial $f\cdot\sum_{l\in L}x_l$ is in the ideal $(x_j^2, j\in\mathbb{N})$. This forces f itself to belong to the latter ideal, as one may easily check. However, this contradicts the fact that $f+(x_i^2, i\in\mathbb{N})$ is a non-zero element of A.

References

[1] D.Eisenbud (2004), *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag.