D-MATH | Commutative Algebra | HS 2018
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Solutions Sheet 6

Associated primes and primary decomposition

Let R be a commutative ring, k an algebraically closed field.

1. Let k be a field. A *monomial ideal* is an ideal $I \subset k[X_1, \dots, X_n]$ generated by monomials in the variables $X_1, \dots, X_n$.

   (a) Characterize those monomial ideals which are prime in $k[X_1, \dots, X_n]$.

   (b) Which monomial ideals are irreducible? Radical? Primary? (Recall that a submodule of a module is called *irreducible* if it cannot be written as the intersection of two larger submodules)

   Solution.

   (a) Assume first that a monomial ideal $I \subset k[X_1, \dots, X_n]$ is prime, and let $X_1^{i_1} \cdots X_n^{i_n}$ a monomial in a fixed finite generating set S of I, where S consists only of monomials. Primality (applied inductively) forces that at least one of the variables X_j effectively appearing in the monomial (i.e., for which $i_j > 0$) belongs to I. Applying this argument with every monomial in S, we obtain a subset $A \subset \{1, \dots, n\}$ such that $X_j \in I$ for all $j \in A$. We claim that $I = (X_j, j \in A)$. By construction of the X_j for $j \in A$, we have that $S \subset (X_j, j \in A)$, whence $I \subset (X_j, j \in A)$. The converse is immediate as well, since we have that $X_j \in I$ for all $j \in A$. We have thus proved that I can be generated by a subset of the set of variables $\{X_1, \dots, X_n\}$.

   Conversely, assume that I has this property, so that $I = (X_j, j \in A)$ for a certain subset $A \subset \{1, \dots, n\}$. We want to prove that I is prime. Let $f, g \in k[X_1, \dots, X_n]$ be such that $f, g \notin I$. Then both f and g have a monomial in which none of the variables $X_j, j \in A$ appears. Therefore, the product fg must also contain a monomial with the same property. In particular, fg cannot belong to I.

   (b) Assume that $I \subset k[X_1, \dots, X_n]$ is an irreducible monomial ideal, and let $S \subset I$ be a minimal finite geneerating set made up of monomials. Assume that one of the monomials in S is of the form $X_1^{i_1} \cdots X_n^{i_n}$ with $i_{j_1}, i_{j_2} > 0$ for some $j_1 \neq j_2$. Then $I = I_1 \cap I_2$ with $I_1 = (S \setminus \{X_1^{i_1} \cdots X_n^{i_n}\}, X_{j_1}^{i_{j_1}})$ and $I_2 = (S \setminus \{X_1^{i_1} \cdots X_n^{i_n}\}, X_1^{i_1} \cdots \widehat{X_{j_1}^{i_{j_1}}} \cdots X_n^{i_n})$, where the notation $X_1^{i_1} \cdots \widehat{X_{j_1}^{i_{j_1}}} \cdots X_n^{i_n}$ means that the power $X_{j_1}^{i_{j_1}}$ is not considered in the product. I_1 and I_2 are strictly larger than I, which is therefore not irreducible.

Assume now that I is generated by powers of some of the variables, i.e. $I = (X_{j_1}^{i_1}, \ldots, X_{j_l}^{i_l})$ for a subset $\{j_1, \ldots, j_l\} \subset \{1, \ldots, n\}$. Assume $I = I_1 \cap I_2$, and choose some finite genrating set S_1, S_2 for I_1, I_2, each made up fo monomials, with $\{X_{j_1}^{i_1}, \ldots, X_{j_r}^{i_r}\} \subset S_1 \cap S_2$. Suppose by contradiction that I_1 and I_2 are strictly bigger that I. Hence there exists a monomial $X_{s_1}^{r_1} \cdots X_{s_m}^{r_m}$ in $S_1 \smallsetminus I$ and a monomial $X_{t_1}^{u_1} \cdots X_{t_k}^{u_k}$ in $S_2 \smallsetminus I$. This means that none of the monomials $X_{j_1}^{i_1}, \ldots, X_{j_l}^{i_l}$ divides the two latter monomials. But then it is clear that the intersection of the ideals $J_1 = (X_{j_1}^{i_1}, \ldots, X_{j_l}^{i_l}, X_{s_1}^{r_1} \cdots X_{s_m}^{r_m})$ and $J_2 = (X_{j_1}^{i_1}, \ldots, X_{j_l}^{i_l}, X_{t_1}^{u_1} \cdots X_{t_k}^{u_k})$ is already strictly bigger than I, and it is contained in $I_1 \cap I_2$, which yields a contradiction.

It is straightforward to verify that a monomial ideal is radical if and only if it is generated by square-free monomials, i.e. monomials of the form $X_{i_1} \cdots X_{i_r}$ for a certain subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$.

Finally, a monomial ideal I is primary if and only if it contains a power of X_j for all j in a given subset of $\{1, \ldots, n\}$ and it is generated by monomials containing no other variables. Indeed, if I is primary, than its radical $\mathrm{Rad}(I)$ is prime. Using the algorithm of the next exercise and the characterization of prime monomial ideals already given in the previous exercise, we immediately get that I contains a power of X_j for some X_j's in $\{X_1, \ldots, X_n\}$, and it has to be generated by monomials containing no other variables. For the converse, it is easy to check that, under the assumption that I contains a power of X_j for some subset of j's in $\{1, \ldots, n\}$ and is generated by monomials containing no other variables, then every zero-divisor in the quotient ring R/I is nilpotent.

2. The setting is the same as in Exercise 1.

   (a) Find an algorithm to compute the radical of a monomial ideal.

   (b) Find an algorithm to compute an irreducible decomposition, and thus a primary decomposition, of a monomial ideal.

Solution.

   (a) Assume S is a minimal finite generating set of a monomial ideal I, S consisting of monomials. Denote by S' the set of all monomials $X_{i_1} \cdots X_{i_r}$, where $X_{i_1}^{j_1} \cdots X_{i_r}^{j_r}$ is a monomial appearing in S. Then we claim that $\mathrm{Rad}(I)$ is the ideal J generated by those monomials. By the previous exercise, J is a radical ideal, since it is generated by square-free monomials. Hence $I \subset J$ implies $\mathrm{Rad}(I) \subset \mathrm{Rad}(J) = J$. On the other hand, if $f \in J$, then some power of f will lie in I: it is sufficient (by the multinomial theorem) to check it for the generators in S', which is immediate.

   (b) As previously, suppose S is a minimal finite generating set of a monomial ideal I, S consisting of monomials. For any monomial $m \in S$, factor it into relatively prime components $m = m_1 \cdots m_k$. Then it is immediate to verify that $I = \bigcap_{i=1}^{k}(I + (m_i))$. Thus, $I = \bigcap_{j=1}^{r} I_j$, where $I_j = (X_{i_j}^{l_j})$, where $X_{i_j}^{l_j}$ is the highest power of X_{i_j} appearing in the m_i (letting m vary over S). By the previous exercise, this is indeed an irreducible decomposition of I.

3. The setting is again the same as in the previous two exercises.

   (a) Let I be the product ideal of the ideals $(X_1), (X_1, X_2), \ldots, (X_1, \ldots, X_n)$. Determine the associated primes of I.

   (b) More generally, for any subset $J \subset \{1, \ldots, n\}$, let $P(J)$ be the prime ideal generated by $\{X_j, j \in J\}$. Let $I_1, \ldots, I_t$ be subsets of $\{1, \ldots, n\}$, and set $I = \prod_{i=1}^{t} P(I_i)$. Let Γ be the "incidence graph", whose vertices are the set $I_i, i = 1, \ldots, t$, with an edge joining I_i and I_j if and only if $I_i \cap I_j \neq \varnothing$. Show that the associated primes of I are precisely those primes that may be expressed as $P(I_{j_1} \cup \cdots \cup I_{j_s})$, where $I_{j_1}, \ldots, I_{j_s}$ forms a connected (i.e., any two vertices can be joined by a finite path made up of edges) subgraph of Γ.

   Solution. There is a detailed solution of this exercise on page 726 of [2].

4. (On the uniqueness of primary decomposition) Let $R = k[X, Y]/I$, where k is a field and $I = (X^2, XY) = (X) \cap (X, Y)^2$. Show that the ideal (Y^n) is (X, Y)-primary in R (considered as a module over $k[X, Y]$), and that

$$0 = (X) \cap (Y^n)$$

   is a minimal primary decomposition of 0 in R for any integer $n \geqslant 1$.

   Solution. We first want to prove that (Y^n) is (X, Y)-primary in R. By a proposition seen in the lecture, it suffices to show that (X, Y) contains the zerodivisors of $R/(Y^n)$ and that $(X, Y)^m \subset \mathrm{Ann}(R/(Y^n))$ for some integer $m \geqslant 1$. Assume that $0 \neq f, g \in R$ are such that $fg \in (Y^n)$, wich means that Y^n divides a product $\tilde{f}\tilde{g}$, where $\tilde{f}, \tilde{g}$ are representatives of f, g in $k[X, Y]$, and Y^n does not divide $\tilde{f}$. This forces $Y|\tilde{g}$, which means $g \in (Y) \subset (X, Y)$. Thus the zerodivisors of $R/(Y^n)$ are contained in (X, Y). Moreover, we have that $(X, Y)^n = (Y^n) \subset \mathrm{Ann}(R/(Y^n))$, which concludes the proof that (Y^n) is (X, Y)-primary in R.

   The ideal $(X) \subset k[X, Y]$ is prime and strictly contains I. Therefore the projected ideal $(X) \subset R$ is a prime ideal of R, which implies $\mathrm{Ass}(R/((X))) = \{(X)\}$. We just proved that (Y^n) is (X, Y)-primary, thus (X) and (Y^n) are both primary ideals in R whose associated primes are not equal. Therefore $0 = (X) \cap (Y^n)$ is a minimal primary decomposition for all $n \geqslant 1$ (the fact that the (X) and (Y^n) have trivial intersection in R is immediate by the definition of R).

5. Let R be an arbitrary commutative ring, $\mathfrak{a} \subset R$ an ideal.

   (a) Suppose that its radical $\mathrm{Rad}(\mathfrak{a})$ is a maximal ideal. Show that $\mathfrak{a}$ is a primary ideal.

(b) Deduce from the previous point that arbitrary powers of maximal ideals are primary ideals.

Solution.

(a) First, $\mathrm{Rad}(\mathfrak{a})$ is a proper ideal, hence $\mathfrak{a}$ is also proper. Then, let $x, y \in R$ be such that $xy \in \mathfrak{a}$ and $x \notin \mathrm{Rad}(\mathfrak{a})$. We need to prove that $y \in \mathfrak{a}$. The ideal $\mathrm{Rad}(\mathfrak{a}) + (x)$ is strictly bigger than $\mathrm{Rad}(\mathfrak{a})$; maximality of the latter forces $R = \mathrm{Rad}(\mathfrak{a}) + (x)$, which means in particular that $1 = z + ax$ for some $z \in \mathrm{Rad}(\mathfrak{a})$ and some $a \in R$. Now $z^n \in \mathfrak{a}$ for some integer $n \geqslant 1$, therefore

$$1 = 1^n = (z + ax)^n = z^n + bx$$

for some element $b \in R$. Thus, $y = z^n y + bxy \in \mathfrak{a}$, being the sum of two elements in $\mathfrak{a}$. The proof is complete.

(b) Let $\mathfrak{m} \subset R$ be a maximal ideal, $n \geqslant 1$ an integer. It is clear that the radical of $\mathfrak{m}^n$ contains $\mathfrak{m}$; moreover, $\mathrm{Rad}(\mathfrak{m}^n)$ is a proper ideal, otherwise $\mathfrak{m}^n$ would be the whole ring. Maximality of $\mathfrak{m}$ forces that $\mathfrak{m} = \mathrm{Rad}(\mathfrak{m}^n)$, whence the latter is a maximal ideal. By the previous point, $\mathfrak{m}^n$ is a primary ideal.

6. Let X be an infinite compact Hausdorff space. Consider the ring $C(X)$ of real-valued continuous functions on X. Is the zero ideal decomposable in this ring?

Solution. We claim that, in $C(X)$, any primary ideal is contained in a *unique* maximal ideal. Indeed, recall that any maximal ideal $\mathfrak{m}$ of $C(X)$ is of the form $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in X$, where $\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$ (Exercise 26, Chapter 1 of [1]). Assume now that $\mathfrak{q} \subset C(X)$ is a primary ideal such that $\mathfrak{q} \subset \mathfrak{m}_x \cap \mathfrak{m}_y$ for some $x \neq y \in X$. Since X is a Hausdorff space, there exists disjoint open neighborhoods V_x, V_y of x and y respectively. As X is also compact, it is a normal topological space, hence by Urysohn's lemma there exist $f, g \in C(X)$ with $f(x) = 1$, $f|_{X \smallsetminus V_x} = 0$, $g(y) = 1$ and $g|_{X \smallsetminus V_y} = 0$. Since V_x and V_y are disjoint, the product fg vanishes everywhere, hence $fg \in \mathfrak{q}$. However, $f \notin \mathfrak{q}$ since $f \notin \mathfrak{m}_x$, and $g^n \notin \mathfrak{q}$ for any integer $n \geqslant 1$ as $g^n \notin \mathfrak{m}_y$. Thus $\mathfrak{q}$ is not primary.

If we know assume that (0) has a primary decomposition, consisting of primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, then we may write $\mathfrak{q}_i \subset \mathfrak{m}_{x_i}$ for some uniquely determined point $x_i \in X$. There exists a function $f_i \in \mathfrak{q}_i$ vanishing at x_i and at no other point of X (otherwise $\mathfrak{q}_i$ would be contained in some $\mathfrak{m}_y$ for $y \neq x_i$). By assumption, X is infinite, hence the product $f = \prod_{i=1}^n f_i$ does not vanish identically on X. On the other hand, $f \in \prod_{i=1}^n \mathfrak{q}_i \subset \bigcap_{i=1}^n \mathfrak{q}_i = (0)$. This contradiction shows that (0) is not decomposable in the ring $C(X)$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.