D-MATH
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Commutative Algebra

HS 2018

Solutions Sheet 7

Primary decomposition and integrality

Let R be a commutative ring, k an algebraically closed field.

1. Let X be a topological space.

   (a) Assume that X is irreducible. Show that any non-empty open subset $O \subset X$ is dense in X and an irreducible topological space when endowed with the subspace topology.

   (b) Assume that $Y \subset X$ is irreducible as a subspace. Show that the closure $\overline{Y}$ in X is also irreducible.

   (c) Show that any irreducible subspace of X is contained in a maximal irreducible subspace.

   (d) Prove that the maximal irreducible subspaces are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

   (e) Let $X = \operatorname{Spec}(R)$, where R is a commutative ring. Prove that the irreducible components of X are the closed sets $V(\mathfrak{p}) = \{\mathfrak{p}' \in \operatorname{Spec}(R) : \mathfrak{p}' \supset \mathfrak{p}\}$, where $\mathfrak{p}$ is a minimal prime ideal of R.

   Solution.

   (a) Let $O \subset X$ be a non-empty open subset. Then we may write $X = \overline{O} \cup (X \smallsetminus O)$ as the union of two closed sets. Since X is irreducible, at least one of them has to be the whole space X. This cannot be the case for $X \smallsetminus O$ because O is assumed to be non-empty. Therefore, $\overline{O} = X$ and thus O is dense.

   Assume now that $O = (O \cap F_1) \cup (O \cap F_2)$ for some closed subsets $F_1, F_2 \subset X$. This in particular gives $O \subset F_1 \cup F_2$, hence $X = (F_1 \cup F_2) \cup (X \smallsetminus O)$, where all elements appearing in this union are closed. Since $X \smallsetminus O$ is a proper closed subset, irreducibility forces $X = F_1 \cup F_2$, which in turn gives that either $F_1 = X$ or $F_2 = X$. As a consequence, either $O = O \cap F_1$ or $O = O \cap F_2$, which yields the desired irreducibility.

   (b) Let $Y \subset X$ be irreducible as a subspace. Assume $\overline{Y} = F_1 \cup F_2$ for some closed subsets $F_1, F_2 \subset X$. Then $Y = (Y \cap F_1) \cup (Y \cap F_2)$, which by irreducibility implies that one of them, say $Y \cap F_1$, is equal to Y. This is equivalent to say that $Y \subset F_1$, but as F_1 is closed, this shows also that $\overline{Y} \subset F_1$, yielding $\overline{Y} = F_1$. The proof is concluded.

(c) Let $Y \subset X$ be irreducible. By Zorn's lemma, it suffices to prove that the non-empty, partially ordered set $\mathcal{P} = \{Y' \subset X \text{ irreducible} : Y \subset Y'\}$ is inductive, i.e. that any totally ordered subset $\mathcal{S} \subset \mathcal{P}$ admits an upper bound. Write $\mathcal{S} = \{Y_\alpha\}_{\alpha \in A}$. If we show that $\tilde{Y} = \bigcup_{\alpha \in A} Y_\alpha$ is irreducible, we have the desired upper bound. Assume $\tilde{Y} = (\tilde{Y} \cap F_1) \cup (\tilde{Y} \cap F_2)$ for some closed subsets $F_1, F_2 \subset X$. Then, by irreducibility of every Y_α, we have that $Y_\alpha \subset F_{i_\alpha}$, with $i_\alpha \in \{1, 2\}$, for all $\alpha \in A$. Since $\mathcal{S}$ is totally ordered, we can actually choose a common i_α for every α, showing that $\tilde{Y} \subset F_{i_0}$ for a certain $i_0 \in \{1, 2\}$. This achieves the result.

(d) Let Y be a maximal irreducible subspace of X. By (b), the closure $\overline{Y}$ is also irreducible. Maximality gives $Y = \overline{Y}$, hence Y is closed.

Points are clearly irreducible subspaces; hence any point is contained in a maximal irreducible subspace by the previous point, which means that maximal irreducible subspaces cover X.

Now assume that X is a Hausdorff space. We claim that any subset consisting of more than one point is not irreducible, which is equivalent to say that the irreducible components of X are points. Since any subspace of a Hausdorff space is Hausdorff, it suffices to prove that an irreducible Hausdorff space X' is a singleton. Any non-empty open subset $O \subset X'$ is dense, hence it intersect any other non-empty open subset $O' \subset X$. If there were two distinct point $x \neq y \in X$, this would contradict the fact that they admit two disjoint neighborhoods.

(e) Let $X = \mathrm{Spec}(R)$. Let us first prove that the irreducible closed subsets of X are the sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal. Now $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, hence it is irreducible, being the closure of a singleton. On the other hand, if $\mathfrak{a}$ is a radical ideal which is not prime, then there are $a, b \in R \smallsetminus \mathfrak{a}$ such that $ab \in \mathfrak{a}$, thus $V(\mathfrak{a}) = V(\mathfrak{a} + (a)) \cup V(\mathfrak{a} + (b))$, showing that $V(\mathfrak{a})$ is not irreducible.

Now clearly minimal primes have to correspond to maximal irreducible subspaces, since the correspondence $\mathfrak{p} \mapsto V(\mathfrak{p})$ is inclusion-reversing.

2. Let R be a commutative ring, and denote by $R[X]$ the ring of polynomials in one indeterminate over R. For any ideal $\mathfrak{a} \subset R$, denote by $\mathfrak{a}[X]$ the set of all polynomials in $R[X]$ with coefficients in $\mathfrak{a}$.

(a) Prove that $\mathfrak{a}[X]$ is the extension of the ideal $\mathfrak{a}$ in $R[X]$.

(b) Prove that, if $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p}[X]$ is a prime ideal in $R[X]$.

(c) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in R, then show that $\mathfrak{q}[X]$ is $\mathfrak{p}$-primary in $R[X]$.

(d) If $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition of $\mathfrak{a}$ in R, then show that $\mathfrak{a}[X] = \bigcap_{i=1}^n \mathfrak{q}_i[X]$ is a minimal primary decomposition of $\mathfrak{a}[X]$ in $R[X]$.

(e) Prove that, if $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$ in R, then $\mathfrak{p}[X]$ is a minimal prime ideal of $\mathfrak{a}[X]$ in $R[X]$.

Solution.

(a) It suffices to remark that $\mathfrak{a}[X]$ is an ideal (which is immediate to check), since obviously $\mathfrak{a}[X] \subset \mathfrak{a}^e$.

(b) Assume $\mathfrak{p}$ is a prime ideal in R. The ideal $\mathfrak{p}[X]$ is the kernel of the canonical surjection $\pi\colon R[X] \to (R/\mathfrak{p})[X]$, hence $R[X]/\mathfrak{p}[X] \simeq (R/\mathfrak{p})[X]$, which is an integral domain since $R/\mathfrak{p}$ is. Thus $\mathfrak{p}[X]$ is a prime ideal.

(c) Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in R. We may write $\mathfrak{q}[X] = \mathfrak{q} + \mathfrak{q} \cdot (x)$, and the same for $\mathfrak{p}[X]$. We now compute the radical of $\mathfrak{q}[X]$: we have

$$\begin{aligned} \operatorname{rad}(\mathfrak{q}[X]) &= \operatorname{rad}(\operatorname{rad}(\mathfrak{q}) + \operatorname{rad}(\mathfrak{q} \cdot (x))) = \operatorname{rad}(\mathfrak{p} + \operatorname{rad}(\mathfrak{q}) \cap \operatorname{rad}((x))) \\ &= \operatorname{rad}(\mathfrak{p} + \mathfrak{p} \cdot (x)) = \operatorname{rad}(\mathfrak{p}[X]) = \mathfrak{p}[X] \ . \end{aligned}$$

It now suffices to prove that $\mathfrak{q}[X]$ is primary. As before, the quotient $R[X]/\mathfrak{q}[X]$ is isomorphic to $(R/\mathfrak{q})[X]$. Since $\mathfrak{q}$ is primary in R, every zerodivisor in $R/\mathfrak{q}$ is nilpotent. Suppose

$$\bar{f}(X) = \sum_{i=0}^{n} \bar{b}_i X^i \in (R/\mathfrak{q})[X]$$

is a zerodivisor. Then, by an exercise of the first sheet, there exists $\bar{a} \in R/\mathfrak{q}$ such that $\bar{a} \cdot \bar{f}(X) = 0$. This implies that any $\bar{b}_i$ is a zerodivisor, hence nilpotent, in $R/\mathfrak{q}$. By the same exercise, we conclude that $\bar{f}(X)$ is nilpotent.

(d) Let $f(X) = \sum_{i=0}^{m} b_i X^i$ be a polynomial in $R[X]$. Then

$$f(X) \in \mathfrak{a}[X] \iff b_i \in \mathfrak{a} \ \forall\, i \iff b_i \in \mathfrak{q}_j \ \forall\, i, j \iff \forall\, j \ f(X) \in \mathfrak{q}_j[X] \ ,$$

so that $\mathfrak{a}[X] = \bigcap_{j=1}^{n} \mathfrak{q}_j[X]$, as desired. By the previous point, each $\mathfrak{q}_j[X]$ is primary, so we have indeed a primary decomposition. Irredundancy of the decomposition follows immediately from the analogous property of the decomposition of $\mathfrak{a}$ and the fact that $\bigcap_{i \neq j} \mathfrak{q}_i[X] = (\bigcap_{i \neq j} \mathfrak{q}_i)[X]$ for all $j = 1, \ldots n$.

(e) Point (b) gives us that $\mathfrak{p}[X]$ is a prime ideal containing $\mathfrak{a}[X]$. Assume now that $\mathfrak{q} \in \operatorname{Spec}(R[X])$ is such that $\mathfrak{a}[X] \subset \mathfrak{q} \subset \mathfrak{p}[X]$; taking contractions, we have $\mathfrak{a} \subset \mathfrak{q} \cap R \subset \mathfrak{p}$, with $\mathfrak{q} \cap R$ prime in R. By minimality of $\mathfrak{p}$ over $\mathfrak{a}$, this forces $\mathfrak{q} \cap R = \mathfrak{p}$, which in turn implies $\mathfrak{q} = \mathfrak{p}[X]$. Therefore, $\mathfrak{p}[X]$ is minimal over $\mathfrak{a}[X]$.

3. The purpose of this exercise is to prove the *Krull intersection theorem*: let R be a noetherian ring, $\mathfrak{a} \subset R$ an ideal, and denote by $\mathfrak{b}$ the intersection of all powers $\mathfrak{a}^n, n \geqslant 1$. Then:

- $\mathfrak{b}\mathfrak{a} = \mathfrak{b}$;

- $\mathfrak{b}(1-a) = (0)$ for some element $a \in \mathfrak{a}$;
- if R is a domain and $\mathfrak{a}$ is a proper ideal, then $\mathfrak{b} = (0)$.

(a) Prove the following preliminary result: if R is a noetherian ring and $I \subset R$ is an ideal, then $(\mathrm{rad}(I))^n \subset I$ for some positive integer n.

(b) Use the previous point to prove the first assertion of Krull's intersection theorem.

(Hint: clearly the non-trivial inclusion is $\mathfrak{b} \subset \mathfrak{ba}$. Prove that $\mathfrak{b} \subset \mathfrak{q}$ whenever $\mathfrak{q}$ is a primary ideal containing $\mathfrak{ba}$ and deduce the result by means of the primary decomposition theorem)

(c) You may admit the following result: if M is a finitely generated R-module, and $I \subset R$ is an ideal such that $IM = M$, then M is annihilated by an element of the form $1-a$, with $a \in I$.

Using this, prove the last two assertions of Krull's intersection theorem.

Solution.

(a) The ideal $\mathrm{rad}(I)$ is generated by finitely many elements $x_1, \ldots, x_r$; for any $i = 1, \ldots r$, there exists an integer $m_i \geqslant 1$ such that $x_i^{m_i} \in I$. Using the multinomial theorem, it is easy to verify that any product of n elements of $\mathrm{rad}(I)$, where $n = \sum_i m_i$, lies in I, so that $(\mathrm{rad}(I))^n \subset I$.

(b) The inclusion $\mathfrak{b} \supset \mathfrak{ba}$ follows from the fact that $\mathfrak{b}$ is an ideal. Since R is noetherian, the ideal $\mathfrak{ba}$ is decomposable, hence it suffices to prove that $\mathfrak{b} \subset \mathfrak{q}$ whenever $\mathfrak{q}$ is a primary ideal containing $\mathfrak{ba}$. For the purpose of contradiction, assume that $\mathfrak{b}$ is not contained is some primary ideal $\mathfrak{q}$ such that $\mathfrak{q} \supset \mathfrak{ba}$. Pick an element $x \in \mathfrak{b} \smallsetminus \mathfrak{q}$; then $xa \in \mathfrak{ba} \subset \mathfrak{q}$ for all $a \in \mathfrak{a}$. Primality of $\mathfrak{q}$ forces $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{q})$. By the previous point, $\mathfrak{a}^n \subset \mathfrak{q}$ for some integer $n \geqslant 1$, which yields $\mathfrak{b} \subset \mathfrak{a}^n \subset \mathfrak{q}$, contradiction.

(c) The ideal $\mathfrak{b}$ is a finitely generated R-module (since R is noetherian). The previous point gives us $\mathfrak{ba} = \mathfrak{b}$, thus, by the admitted result, $\mathfrak{b}$ is annihilated by an element of the form $1-a$, with $a \in \mathfrak{a}$. This proves the second assertion of the theorem.

If $\mathfrak{a}$ is proper, then $1 \notin \mathfrak{a}$, hence $1 - a \neq 0$ for all $a \in \mathfrak{a}$. Since R is a domain, the second point of the theorem implies that $\mathfrak{b} = (0)$.

4. Let $\varphi\colon R \to S$ be an integral homomorphism of rings. Show that the induced continuous map $\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed, namely that it maps closed sets to closed sets.

Solution. By definition of an integral homomorphism, S in integral over the subring $S' = \varphi(R)$. We write φ as the composition of $\varphi'\colon R \to S'$ and the inclusion $i\colon S' \to S$. It suffices to prove that both $(\varphi')^*$ and i^* are closed maps.

By an exercise in the previous sheets, $(\varphi')^*$ induces an homeomorphism between $\operatorname{Spec}(S')$ and $V(\ker\varphi) \subset \operatorname{Spec}(R)$ (since φ' is surjective). $V(\ker\varphi)$ being closed in $\operatorname{Spec}(R)$, $(\varphi')^*$ maps closed sets of $\operatorname{Spec}(S')$ to closed sets of $\operatorname{Spec}(R)$.

It remains to prove that i^* is a closed map. Let $K \subset \operatorname{Spec}(S)$ be closed, which means that there exists a radical ideal $\mathfrak{b} \subset S$ such that $K = V(\mathfrak{b})$. We claim that $i^*(K) = V(\mathfrak{b} \cap S')$, so that in particular it is closed in $\operatorname{Spec}(S')$. If $\mathfrak{q} \supset \mathfrak{b}$ is prime in S, then $\mathfrak{q} \cap S' \supset \mathfrak{b} \cap S'$, thus $i^*(K) \subset V(\mathfrak{b} \cap S')$.

Now let $\mathfrak{p} \supset \mathfrak{b} \cap S'$, and denote by $\overline{\mathfrak{p}}$ its image in $S'/\mathfrak{b} \cap S'$ under the projection map. The induced ting homomorphism $\eta\colon S'/S' \cap \mathfrak{b} \to S/\mathfrak{b}$ is integral, so by theorem 5.10 in [1] there exists $\overline{\mathfrak{q}} \in \operatorname{Spec}(S/\mathfrak{b})$ such that $\eta^*(\overline{\mathfrak{q}}) = \overline{\mathfrak{p}}$. The conclusion is achieved recalling that there are canonical homeomorphisms $\operatorname{Spec}(S/\mathfrak{b}) \simeq V(\mathfrak{b})$ and $\operatorname{Spec}(S'/\mathfrak{b} \cap S') \simeq V(\mathfrak{b} \cap S')$.

5. Let R be a subring of a ring S such that S is integral over R, and let $\varphi\colon R \to \Omega$ be a ring homomorphism of A into an algebraically closed field Ω. Show that φ admits an extension to a ring homomorphism $\Phi\colon S \to \Omega$.

   (Hint: use theorem 5.10 in [1])

   Solution. Ω is an integral domain, hence $\varphi(R)$ is a domain as well. This implies that $\mathfrak{p} = \ker\varphi$ is a prime ideal in R. By theorem 5.10 in [1], there is a prime ideal $\mathfrak{q} \subset S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then $S/\mathfrak{q}$ is integral over $R/\mathfrak{p}$; therefore, it is sufficient to prove the result when R and S are integral domains and $\varphi\colon R \to \Omega$ is an injective map.

   Apply Zorn's lemma to show that there is a maximal extension $\Phi\colon S' \to \Omega$ of φ to a subring $R \subset S' \subset S$. Our aim is to show that $S' = S$. By contradiction, assume there exists $x \in S \setminus S'$. Since x is integral over R, it is *a fortiori* integral over S'; let $f(X) \in S'[X]$ be a monic polynomial such that $f(x) = 0$. Let $\tilde{f}(X)$ be the canonical image of $f(X)$ in $\Omega(X)$. The composed map

   $$S'[X] \to \Omega[X] \twoheadrightarrow \Omega[X]/(\tilde{f}(X))$$

   factors through an injection $S'[x] \simeq S'[X]/(f(X)) \to \Omega[X]/(\tilde{f}(X))$, which clearly extends Φ. Now Ω is algebraically closed, hence $\Omega[X]/(\tilde{f}(X)) \simeq \Omega$, being an algebraic extension of Ω. This contradicts maximality of S'.

6. Let G be a finite group of automorphisms of a ring R, and let R^G denote the subset of G-invariant elements, i.e. $R^G = \{x \in R : \sigma(x) = x \text{ for all } \sigma \in G\}$.

   (a) Prove that R^G is a subring of R and that R is integral over R^G.

   (b) Suppose $S \subset R$ is a multiplicatively closed subset such that $\sigma(S) \subset S$ for all $\sigma \in G$, and denote by $S^G = S \cap R^G$. Show that the action of G on R extends to an action on $R[S^{-1}]$, and prove that

   $$R^G[(S^G)^{-1}] \simeq (R[S^{-1}])^G\ .$$

(c) Let $\mathfrak{p}$ be a prime ideal of R^G, and let P the set of prime ideals of R whose contraction is $\mathfrak{p}$. Show that G acts transitively on P and deduce that P is a finite set.

Solution.

(a) The fact that R^G is a subring of R is clear since each $\sigma \in G$ is a ring automorphism. Let $x \in R$, and consider the monic polynomial

$$f(T) = \prod_{\sigma \in G} (T - \sigma(x)) \in R[T] \ .$$

It is clear that x is a root of f; moreover, for each $\sigma_0 \in G$,

$$\sigma_0(f(T)) = \prod_{\sigma \in G} (T - \sigma_0(\sigma(x))) = \prod_{\tau \in G} (T - \tau(x)) = f(T)$$

since composition with σ_0 induces a permutation of G. We deduce that all the coefficients of $f(T)$ are G-invariant, namely $f(T) \in R^G(T)$. Thus, x is integral over R^G.

(b) We need to extend each $\sigma \in G$ to an automorphism $\overline{\sigma}$ of $R[S^{-1}]$. The only possible way to do this is to define $\overline{\sigma}(xs^{-1}) = \sigma(x)\sigma(s)^{-1}$ for all element xs^{-1} of $R[S^{-1}]$. This is well defined as a map from $R[S^{-1}]$ to itself, as S is invariant under G, and it is immediate to check that it is a ring homomorphism extending σ to $R[S^{-1}]$. Since we can likewise construct $\overline{\sigma^{-1}}$, and since obviously $\overline{\sigma^{-1}} \circ \overline{\sigma} = \overline{\sigma^{-1} \circ \sigma} = \overline{\mathrm{id}_R} = \mathrm{id}_{R[S^{-1}]}$, we get that $\overline{\sigma}$ is an automorphism of $R[S^1]$. In this way, we have extended the action of G on R to an action on $R[S^{-1}]$.

The canonical ring homomorphism $R^G \hookrightarrow R \to R[S^{-1}]$ factors through a morphism $\chi\colon R^G \to (R[S^{-1}])^G$; each element of S^G gets mapped to an invertible element of $(R[S^{-1}])^G$ by χ, thus the universal property of localization gives the existence of a ring map $\tilde{\chi}\colon R^G[(S^G)^{-1}] \to (R[S^{-1}])^G$. We claim that $\tilde{\chi}$ is bijective, so that we get the desired isomorphism.

First, assume $xs^{-1} = 0$ in $(R[S^{-1}])^G$. Then, there exists $y \in S$ such that $yx = 0$. Taking $y' = \prod_{\sigma \in G} \sigma(y)$, we also get $y'x = 0$. This means however that $xs^{-1} = 0$ already in $R^G[(S^G)^{-1}]$; thus, $\tilde{X}$ is bijective.

If xs^{-1} is an arbitrary element of $R[S^{-1}]^G$, then consider $s' = \prod_{\sigma \in G \setminus \{\mathrm{id}_G\}}$. Since both xs^{-1} and ss' are G-invariant, then so is their product xs'. Hene, for any $\sigma \in G$, there exists $t_\sigma \in S$ such that $t_\sigma xs' = t_\sigma \sigma(xs')$. Define $t = \prod_{\tau \in G} \tau(\prod_{\sigma \in G} t_\sigma)$. We have $txs' \in R^G$, thus $xs^{-1} = (ts'x)(ts's)^{-1}$, with the numerator belonging to R^G. The proof is concluded.

(c) Let $\mathfrak{q}, \mathfrak{q}'$ be elements of P. If $x \in \mathfrak{q}$, then $\prod_{\sigma \in G} \sigma^{-1}(x) \in R^G \cap \mathfrak{q} = \mathfrak{p}$. Since also $\mathfrak{p} = R \cap \mathfrak{q}'$, we get $\prod_{\sigma \in G} \sigma^{-1}(x) \in \mathfrak{q}'$. Primality of $\mathfrak{q}'$ implies that there exists some element $\sigma \in G$ such that $y = \sigma^{-1}(x) \in \mathfrak{q}'$. Hence, $x = \sigma(y) \in \sigma(\mathfrak{q}')$. As x was arbitrary in $\mathfrak{q}$, we deduce that $\mathfrak{q} \subset \bigcup_{\sigma \in G} \sigma(\mathfrak{q}')$. Primality forces again $\mathfrak{q} \subset \sigma(\mathfrak{q}')$. Now the fact that $\mathfrak{q} \cap R^G = \mathfrak{p}$ and $\sigma(\mathfrak{q}') \cap R^G = \sigma(\mathfrak{q}') \cap \sigma(R^G) = \sigma(\mathfrak{q}' \cap R^G) = \sigma(\mathfrak{p}) = \mathfrak{p}$, corollary 5.9 in [1] gives us that actually $\mathfrak{q} = \sigma(\mathfrak{q}')$. We have thus shown that G acts transitively on P; since G is finite, P must then be also finite.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.