D-MATH Commutative Algebra HS 2018
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Solutions Sheet 9

Integrality, Jacobson rings and Krull topology

Let R be a commutative ring, k an algebraically closed field.

1. let R, S be commutative rings. A ring homomorphism $f\colon R \to S$ is said to have the *going-up property* if the conclusion of the going-up theorem 5.11 in [1] holds for S and its subring $f(R)$.

   Let $f^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the mapping associated with f.

   Consider the following three statements:

   (a) f^* is a closed mapping;

   (b) f has the going-up property;

   (c) let $\mathfrak{q}$ be any prime ideal of S and $\mathfrak{p}$ the contraction of $\mathfrak{q}$ in R. Then $f^*\colon \mathrm{Spec}(S/\mathfrak{q}) \to \mathrm{Spec}(R/\mathfrak{p})$ is surjective.

   Prove that (a) implies (b), and that (b) is equivalent to (c).

   Solution. The map f factors as $f = i \circ p$, where p is surjective and $i\colon f(R) \to S$ is the canonical inclusion. We know for previous exercise sheets that the induced map p^* maps $\mathrm{Spec}(f(R))$ homeomorphically onto the closed subset $V(\ker f) \subset \mathrm{Spec}(R)$. For any $\mathfrak{p} \in V(\ker f)$, surjectivity of p implies that $p(R)/p(\mathfrak{p}) \simeq R/\mathfrak{p}$, so that f will satisfy any of the three properties if and only if i does. We can thus assume without loss of generality that $f\colon R \hookrightarrow S$ is an inclusion.

   It is also straightforward that, under the assumption that f is an inclusion, the going-up property is equivalent to the following: for any $\mathfrak{p} \in \mathrm{Spec}(R)$ and any $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{p} = \mathfrak{q} \cap R$, the restriction map $f^*|_{V(\mathfrak{q})}\colon V(\mathfrak{q}) \to V(\mathfrak{p})$ is surjective.

   $[(a) \Rightarrow (b)]$ Assume f^* is a closed mapping, and let $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Since $V(\mathfrak{q})$ is closed in $\mathrm{Spec}(S)$, by assumption $f^*(V(\mathfrak{q}))$ is closed in $\mathrm{Spec}(R)$ and contains $\mathfrak{p}$, so that $\overline{\mathfrak{p}} \subset f^*(V(\mathfrak{q}))$. Since clearly $\overline{\mathfrak{p}} = V(\mathfrak{p})$, we deduce that $f^*|_{V(\mathfrak{q})}\colon V(\mathfrak{q}) \to V(\mathfrak{p})$ is surjective.

   $[(b) \Leftrightarrow (c)]$ Resorting to an exercise of previous sheets, we may identify the map $f^*|_{V(\mathfrak{q})}\colon V(\mathfrak{q}) \to V(\mathfrak{p})$ with the map $\bar{f}^*\colon \mathrm{Spec}(S/\mathfrak{q}) \to \mathrm{Spec}(R/\mathfrak{p})$ induced by the map $\bar{f}\colon R/\mathfrak{p} \to S/\mathfrak{q}$. It is then obvious that the two properties are equivalent.

2. The aim of this exercise is to prove *Noether's normalization lemma*: let k be an infinite field, $A \neq 0$ a finitely generated k-algebra. Then there exist elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \dots, y_r]$.

   (a) Prove that there are generators $x_1, \dots, x_n$ of A as a k-algebra such that $x_1, \dots, x_r$ are algebraically independent over k and each of $x_{r+1}, \dots, x_n$ is algebraic over $k[x_1, \dots, x_r]$, for some $0 \leqslant r \leqslant n$.

   (b) Argue by induction on n (if $n = r$ there is nothing to do). Suppose $n > r$ and the result true for $n-1$ generators.

      i. Show that there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \dots, x_{n-1}, x_n) = 0$.

      ii. Let F be the homogeneous part of highest degree of f. Use the assumption that k is infinite to show that there exist $\lambda_1, \dots, \lambda_{n-1} \in k$ with $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$.

      iii. Set $x_i' = x_i - \lambda_i x_n$ for all $1 \leqslant i \leqslant n-1$. Prove that x_n is integral over the ring $A' = k[x_1', \dots, x_{n-1}']$, and conclude that A is integral over A'.

   (c) Apply the inductive hypothesis to conclude the proof.

Solution.

   (a) Let $x_1, \dots, x_n$ be generators of A as a k-algebra. Define $S \subset \{x_1, \dots, x_n\}$ to be a maximal subset of elements which are algebraically independent over k (no Zorn's lemma needed, we are working over a finite set). We may renumber the x_i so that $S = \{x_1, \dots, x_r\}$ for some $0 \leqslant r \leqslant n$ (adopting the convention that $S = \varnothing$ if $r = 0$). Then each of the x_j for $j > r$ has to be algebraic over $k[x_1, \dots, x_r]$ (as in the proof if the existence of a transcendence basis), since otherwise the subset $\{x_1, \dots, x_r, x_j\}$ would be algebraically independent over k, contradicting the maximality of S.

   (b) Suppose $n > r$ and that the result is true for $n-1$ generators.

      i. Since the element x_n is algebraic over $k[x_1, \dots, x_{n-1}]$ there exists a polynomial $0 \neq f' \in k[x_1, \dots, x_{n-1}][X]$ such that $f'(x_n) = 0$. We may clearly regard f' as a polynomial with coefficients in k and indeterminates $X_1, \dots, X_n$, so that the conclusion is reached.

      ii. Suppose that such $\lambda_i, 1 \leqslant i \leqslant n-1$, don't exist. Then we have, by homogeneity of F, $F(X_1, \dots, X_n) = X_n^{\deg F} F(X_1, \dots, X_{n-1}, 1)$; now, since k is an infinite field, the polynomial $F(X_1, \dots, X_{n-1}, 1)$ vanishes in the ring $k[X_1, \dots, X_{n-1}]$ (otherwise the λ_i would exist). Hence we deduce $F = 0$, contradicting the fact that $f \neq 0$.

iii. Define $x'_i = x_i - \lambda_i x_n$ for all $1 \leqslant i \leqslant n-1$. We may write

$$F(x_1, \dots, x_n) = \sum_I a_I \prod_{j=1}^{n} x_j^{i_j} = \sum_I a_I x_n^{i_n} \prod_{j=1}^{n-1} (x'_j + \lambda_j x_n)^{i_j} \ .$$

The coefficient of $x_n^{\deg F}$ in $F \in k[x'_1, \dots, x'_{n-1}, x_n]$ is thus

$$c = \sum_I a_I \lambda_1^{i_1} \cdots \lambda_{n-1}^{i_{n-1}} = F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0 \ ,$$

so that the equation $c^{-1} f(x_1, \dots, x_{n-1}, x_n) = 0$ is monic when written in $A'[x_n]$. As a consequence, x_n is integral over A'. This implies that $A = k[x_1, \dots, x_n]$ is integral over A'. Using the induction hypothesis, A' is integral over some $k[y_1, \dots, y_r]$, with $y_1, \dots, y_r$ algebraically independent over k. By transitivity of integral dependence, A is integral over $k[y_1, \dots, y_r]$.

3. Let k be an algebraically closed field.

   (a) Prove that k is infinite.

   (b) Let X be an affine algebraic variety in k^n with coordinate ring $A \neq 0$. Use the outlined proof of Noether's normalization lemma to prove that there exists a linear subspace L of dimension r in k^n and a linear mapping $k^n \to L$ which maps X onto L.

Solution.

(a) Assume that k is a finite field, say $k = \{a_1, \dots, a_n\}$. Then it is obvious that the non-constant polynomial $f(X) = \prod_{i=1}^{n}(X - a_i) + 1 \in k[X]$ has no roots over k. Thus k is not algebraically closed.

(b) Let X be an affine variety in k^n defined by an ideal $I(X)$ such that its coordinate ring $A = k[X_1, \dots, X_n]/I(X)$ is non-zero. A is obviously a finitely generated k-algebra. Since k is infinite by the previous point, Noether's normalization lemma applies and gives elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \dots, y_r]$. The proof given in the previous exercise shows that the y_i may be choosen to be linear combinations of the $x_1, \dots, x_n$ (where here x_i denotes the projection of X_i onto A).

Denote by $\iota \colon X \to k^n$ the canonical inclusion; we want to find a linear subspace L of dimension r in k^n and a linear map $\varphi \colon k^n \to L$ such that $\varphi \circ \iota$ is surjective. Transposing this condition in the language of the associated coordinate rings, this is equivalent to find linear polynomials $f_1, \dots, f_{n-r} \in k[X_1, \dots, X_n]$ and a "linear" map $\varphi^* \colon k[X_1, \dots, X_n]/(f_1, \dots, f_{n-r}) \to k[X_1, \dots, X_n]$

such that the map $\pi \circ \varphi^* \colon k[X_1, \ldots, X_n]/(f_1, \ldots, f_{n-r}) \to A$ is injective, where $\pi \colon k[X_1, \ldots, X_n] \to A$ denotes the canonical projection. We are thus lead to find a linear function $\psi \colon k[X_1, \ldots, X_n] \to A$ whose kernel is an ideal generated ny $n - r$ linear polynomials $f_1, \ldots, f_{n-r}$. It suffices to express the $X_j, j = 1, \ldots, n$ (actually an appropriate subset of them) linearly as functions of the $y_i, 1 \leqslant i \leqslant r$, and send each X_j to the thus-obtained linear combination of the y_i.

4. Let R be a ring. Show that the following are equivalent:

   (a) R is a Jacobson ring;

   (b) for any ring S and any surjective ring homomorphism $f \colon R \to S$, the nilradical ideal of S coincides with its Jacobson ideal;

   (c) every prime ideal in R which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

   (*Hint: for the implication* (*c*) $\Rightarrow$ (*a*) *argue as follows. Assume* (*a*) *is false, then there is a prime ideal which is not the intersection of maximal ideals. Passing to the quotient ring, we may assume that R is a domain with non-zero Jacobson ideal. Pick a non-zero f in the Jacobson ideal, then $R_f \neq 0$, thus R_f has a maximal ideal whose contraction in R is a prime ideal not containing f, and which is maximal with respect to this property.*)

   Solution. [(*a*) $\Rightarrow$ (*b*)] Let $\mathfrak{a} \subset R$ be an ideal, and denote by $M(\mathfrak{a})$ the set of maximal ideals of R containing $\mathfrak{a}$. By assumption the radical ideal $\mathrm{rad}(\mathfrak{a})$ is the intersection of all elements of $M(\mathfrak{a})$. This clearly implies that, in the quotient $R/\mathfrak{a}$, the Jacobson and the nilradical ideal coincide. Since any homomorphic image of R is isomorphic to $R/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset R$ by the first isomorphism theorem for rings, the conclusion is achieved.

   [(*b*) $\Rightarrow$ (*c*)] Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be not maximal. Then the trivial ideal (0) is not maximal in the quotient $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is an integral domain, its nilradical vanishes; by assumption, the nilradical coincides with the Jacobson radical, which therefore vanishes as well. Going back to the ring R and the ideal $\mathfrak{p}$, this implies precisely that $\mathfrak{p}$ is the intersection of all prime ideals containing it strictly.

   [(*c*) $\Rightarrow$ (*a*)] We follow the given hint, thus in particular we may assume that R is a domain with non-zero Jacobson ideal. Choose a non-zero $f \in \mathrm{Jac}(R)$, so that the localization $R_f \neq 0$. Now R_f contains a maximal ideal $\mathfrak{q}$, whose contraction $\mathfrak{q}'$ is a prime ideal which is maximal with respect to the property of not meeeting S_f. By assumption $\mathfrak{q}'$ is an intersection of prime ideals containing f, which gives the desired contradiction.

5. Let G be a group (not necessarily abelian), and let $\mathcal{F}$ be a filter on G satisfying the following properties:

- for any $V \in \mathcal{F}$ there exists $U \in \mathcal{F}$ such that $UU^{-1} \subset V$, where $UU^{-1} = \{xy^{-1} : x, y \in U\}$;
- for any $V \in \mathcal{F}$ and any $g \in G$, $gVg^{-1} \in \mathcal{F}$, where $gVg^{-1} = \{gxg^{-1} : x \in V\}$.

Prove that there exists a unique topology τ on G making G into a topological group (i.e. multiplication and inverse are continuous maps with respect to τ) and for which $\mathcal{F}$ is the filter of neighborhoods of the identity element e_G.

Solution. We first show uniqueness. If τ is a topology on G making it into a topological group, then for any fixed $a \in G$ the map $G \ni g \mapsto ag$ is an homeomorphism. Thus, the filter of neighborhoods $\mathcal{F}_a$ of a is simply the a-translated of $\mathcal{F}$, assuming that τ induces the filter $\mathcal{F}$ as filter of neighborhoods of the identity. Therefore, $\mathcal{F}$ uniquely determines the filter of neighborhoods of every point, and thus the topology (two topologies on a set having the same filters of neighborhoods at every point must coincide).

Existence is more involved. The first property clearly implies that $e_G \in V$ for any $V \in \mathcal{F}$. Now let $g \in G$, and define $\mathcal{F}_g = \{gV : V \in \mathcal{F}\}$, the g-translated of $\mathcal{F}$. It is immediate that $\mathcal{F}_g$ is a filter on G all of whose elements contain the point g. We now claim that, for any gV in $\mathcal{F}_g$, there exists $gW \in \mathcal{F}_g$ such that $gV \in \mathcal{F}_h$ for all $h \in gW$. This, by a general result in general topology, implies that there exists a topology τ for which $\mathcal{F}_g$ is the filter of neighborhoods at g, for any $g \in G$. The first property of $\mathcal{F}$ implies that, given $V \in \mathcal{F}$, there exists $W \in \mathcal{F}$ such that $WW \subset V$; indeed, take $U \in \mathcal{F}$ such that $UU^{-1} \subset V$. Then any $W \in \mathcal{F}$ with $W \subset U \cap U^{-1}$ will verify the property. We now claim that, with this choice of W, we have that $gV \in \mathcal{F}_h$ for all $h \in gW$. In fact, if $h \in gW$, then $hW \in \mathcal{F}_h$ and $hW \subset gWW \subset gV$, so that $gV \in \mathcal{F}_h$.

We now prove that the group operations are continuous with respect to τ. It clearly suffices to show that the map $G \times G \ni (g, h) \mapsto gh^{-1}$ is continuous. Fix $(g_0, h_0) \in G \times G$, and let $g_0h_0^{-1}V$, with $V \in \mathcal{F}$, be an arbitrary neighborhood of $g_0h_0^{-1}$. By the assumption on $\mathcal{F}$, there exists $U \in \mathcal{F}$ such that $UU^{-1} \subset h_0^{-1}Vh_0$. Thus $(g_0h_0^{-1})^{-1}[g_0U(h_0U)^{-1}] = h_0g_0^{-1}g_0UU^{-1}h_0^{-1} = h_0UU^{-1}h_0^{-1} \subset h_0(h_0^{-1}Vh_0)h_0^{-1} = V$, which means that $g_0U(h_0U)^{-1} \subset g_0h_0^{-1}V$,showing the desired continuity at the point $(g_0, h_0) \in G \times G$.

6. Let G be an abelian group, and let $G = G_0 \supset G_1 \supset \cdots$ be a descending filtration of subgroups. Consider the set $\mathcal{F} = \{V \subset G : G_n \subset V \text{ for some } n \in \mathbb{N}\}$

   (a) Prove that $\mathcal{F}$ satisfies the conditions in the previous exercise. The resulting topology on G is called the *Krull topology* (determined by the given filtration).

   (b) Show that if $H \in \mathcal{F}$ is a subgroup, then it is open for the Krull topoogy.

   (c) Assume that $G = R$ is a commutative ring, and suppose the G_i are ideals of R. Prove that ring multiplication is continuous with respect to the Krull topology (so that R becomes a *topological ring*).

Solution.

(a) We first show that $\mathcal{F}$ is a filter. The set $\mathcal{F}$ is non-empty by definition. Moreover, each $V \in \mathcal{F}$ is non-empty because it contains $G_n \neq \varnothing$ for some $n \in \mathbb{N}$. Clearly, if $V \in \mathcal{F}$ and $W \supset V$, then $W \in \mathcal{F}$ as well. Finally, if $V, W \in \mathcal{F}$, pick integers n, m such that $G_n \subset V$ and $G_m \subset W$; then $G_{max\{n,m\}} \subset V \cap W$, thus $V \cap W \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter.

The second property is obviously satisfied since G is abelian. For the first one, simply notice that $G_n G_n^{-1} = G_n$ for all n, since G_n is a subgroup. Thus, $\mathcal{F}$ verifies all the assumptions of the previous exercise; as a consequence, there is a well-defined group topology on G for which $\mathcal{F}$ is the filter of neighborhoods of the identity.

(b) Let $H \in \mathcal{F}$ be a subgroup, so that in particular $G_n \subset H$ for some $n \in \mathcal{N}$. For any point $h \in H$, the set hG_n is a neighborhood of h; moreover, $hG_n \subset H$ since H is a subgroup, so that H is also a neighborhood of h. Thus H is a neighborhood of any of its elements, hence it is open.

(c) Fix $x, y \in R$, and let xyG_i be an arbitrary basis neighborhood of the point xy for the Krull topology. Then $(xG_i)(yG_i) = xyG_iG_i \subset xyG_i$, as multiplication is commutative and G_i is an ideal. Thus, $xG_i \times yG_i$ is a neighborhood of $(x, y) \in R \times R$ for the product topology which is mapped by the multiplication operation inside xyG_i. This shows continuity of multiplication at an arbitrary point $(x, y) \in R \times R$.

References

[1] M.Atiyah, Y.McDonald (1994), *Introduction to commutative algebra*, Addison-Wesley Publishing Company.

[2] D.Eisenbud (2004), *Commutative Algebra with a View towards Algebraic Geometry*, Springer.