Objectives for commutative algebra, Fall 2018

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Contents

1 Overview

Here I indicate the main things you should be able to do after each lecture, and refer to the references:

- [A-M] = Atiyah–Macdonald
- [Eis] = Eisenbud
- [Bosch] = Bosch,
- [Mat] = Matsumura

2 18 Sep, first half: Introduction

Objectives. You should have some sense of what this course in commutative algebra will cover:

- fundamental techniques such as localization, completion, ...,
- integral extensions of rings,
- factorizations of ideals, Dedekind domains,
- primary decomposition,
- dimension theory of rings,
- etc.

You should have some hint of why such things might be useful in algebraic geometry and algebraic number theory.

References.

- [Eis, Ch. 0]
- skim [Eis, Ch. 1] for some history/motivation
- [A-M, p1-5] (up to "Nilradical and Jacobson radical")

3 18 Sep, second half: Can every variety be defined using only finitely many equations?

Objectives. You should be able to answer this question affirmatively by reducing it to the Hilbert basis theorem and proving the latter.

References.

- Eis 1.4
- secondary: A-M p51

4 20 Sep: When do two systems of polynomial equations have the same solutions?

Objectives. You should be able to state the Nullstellensatz, illustrate its basic content, and explain how it answers the titled question over an algebraically closed base field.

References.

- Eis 1.6 (algebra, geometry, Nullstellensatz)

5 25 Sep: Equivalent forms of the Nullstellensatz; localization

Objectives. You should be able to state and relate several equivalent forms of the weak Nullstellensatz. You should be able to explain how it implies the strong Nullstellensatz via the trick of adding a variable. You should be able to construct the localization of a ring with respect to a multiplicatively closed set and describe the homomorphisms from that localization to another ring.

References.

- Eis 1.6 (algebra, geometry, Nullstellensatz)
- Eis 2, intro and 2.1 (fractions)
- A-M Prop 3.11 (extended and contracted ideals in rings of fractions)

6 27 Sep: Localization vs. ideal theory

Objectives. You should be able to describe the ideal theory of a localization in terms of that of the original ring. You should be able to apply this description to deduce certain basic facts, such as the implication from the weak to the strong Nullstellensatz and the equality between the radical of an ideal and the intersection of the primes containing it.

References.

- Eis 2.1, intro and 2.1 (fractions)
- Eis 2.3 (construction of primes)
- secondary: A-M 3

7 2 Oct: Primes and irreducible varieties; modules

Objectives. You should be able to explain how prime ideals correspond to irreducible varieties. You should be able to define a module and give some basic examples: over the integers, over a field, free modules, etc.

References.

- A-M Ch1 Exercises
- A-M p17-20, Eis 0.3 (preliminaries on modules)

8 4 Oct: Hom and exactness, generators and relations; tensor product definition

Objectives. You should be able to explain the basic exactness properties of Hom, or equivalently, the kernels of the pullback and pushforward morphisms between Hom spaces. You should be able to explain how to define a module using generators and relations as a quotient of a free module, and then to describe the morphisms out of such a module.

References.

- Eis 2.2 (Hom and tensor)
- A-M p22-31 (exact sequences, tensors)

9 9/11 Oct: Tensor product, flatness, local properties, Noetherian modules

Objectives. You should be able to define the tensor product, explain its basic exactness properties, and define "flatness". A decent test is whether you understand in full detail the proof that Hom commutes with flat base extension for finitely-presented modules.

References.

- Eis 2.2 (Hom and tensor)
- A-M p22-31 (exact sequences, tensors)

10 16/18 Oct: Associated primes, primary ideals and modules

Objectives. You should be define and study basic properties of the set of associated primes of a finitely-generated module over a Noetherian ring: nonemptyness, finiteness, minimal/isolated vs. embedded, etc. You should be able to describe the special case $M = R/I$ for a radical ideal I. You should be able to relate several equivalent definitions of (co-)primary modules.

References.

- Eis 3.1-3.3; see also A-M 4

11 23/25 Oct: Primary decompositions

References.

- Eis 3.1-3.3, A-M 4

12 30 Oct:

Objectives. You should be able to apply the Cayley–Hamilton theorem to prove Nakayama's lemma as well as basic properties of integral ring extensions.

References.

- Eis 4.1, A-M 5

13 1 Nov:

Objectives. Further basics on integral ring extensions, behavior of primes.
References.

- Eis 4.1, 4.3, 4.4; A-M 5

14 6-15 Nov

Objectives. Nullstellensatz, Artin–Rees, Tor
References.

- Eis 4, 5, 6

15 20/22 Nov

Objectives. Completions
References.

- Eis 7

16 27 Nov

Objectives. More on flatness and Tor
References.

- Eis 6

17 29 Nov

Objectives. Artin rings
References.

- A-M 8

18 4/6 Dec

Objectives. Krull's principal ideal theorems; dimension of polynomial rings
References.

- Bosch 2.4
- Eis 10, 13

19 11 Dec

Objectives. Valuation rings
References.

- A-M 5, 10

20 13 Dec

Objectives. DVR's, Dedekind domains
References.

- A-M 5, 9
- Eis 11