Exercise Sheet 1

Exercise 1

Let G_α be a topological group, where $\alpha \in A$ is a family of indices. Show that the product group $\prod_{\alpha \in A} G_\alpha$ endowed with the product topology is a topological group as well.

Exercise 2

Show that the topological group $O(p,q)$ for $p,q \geq 1$ is not compact.

Exercise 3

Let p be a prime number. Prove that the map

$$i : \mathbb{Z} \to \mathbb{Z}_p$$

given by $i(x) = (x \mod p^n)_{n \in \mathbb{N}}$ is injective with dense image.

Exercise 4

Let (X,d) be a metric space. Suppose that the closed ball $B_{\leq r}(x) = \{y \in X | d(x,y) \leq r\}$ of radius r centered at x is compact, for all $r \geq 1$ and all $x \in X$. Show the set $\mathrm{Isom}(X)$ of the isometries of X is a locally compact topological group when endowed with the compact-open topology.

Exercise 5

Let G be a topological group and let V be a neighborhood of the neutral element e. Show that there exists an open set W such that $e \in W$, $W^2 \subset V$ and $W = W^{-1}$.