Solutions: Exercise Sheet 1

Exercise 1

Let G_α be a topological group, where $\alpha \in A$ is a family of indices. Show that the product group $\prod_{\alpha \in A} G_\alpha$ endowed with the product topology is a topological group as well.

Solution: Since G_α is a topological group, we know that for each $\alpha \in A$ the maps

$$\mu_\alpha : G_\alpha \times G_\alpha \to G_\alpha, \quad i_\alpha : G_\alpha \to G_\alpha$$

of multiplication and inversion are continuous. If we set $G := \prod_{\alpha \in A} G_\alpha$ and we denote an element $g \in G$ as $g = (g_\alpha)_{\alpha \in A}$, we know that the multiplication and the inversion for G are defined as it follows

$$\mu : G \times G \to G, \quad \mu((g_\alpha)_{\alpha \in A}, (h_\alpha)_{\alpha \in A}) := (\mu_\alpha(g_\alpha, h_\alpha))_{\alpha \in A},$$
$$i : G \to G, \quad i((g_\alpha)_{\alpha \in A}) := (i_\alpha(g_\alpha))_{\alpha \in A}.$$

We need to show that both μ and i are continuous. We are going to prove the statement only for i, since the procedure is similar in the case of the mulplication. Denote by $p_\alpha : G \to G_\alpha$ the projection map with respect to the α-component, that means $p((g_\alpha)_{\alpha \in A}) := g_\alpha$. By the universal property of the product topology, we know that i is continuous if and only if $p_\alpha \circ i$ is continuous for every $\alpha \in A$.

Let now $V_\alpha \subset G_\alpha$ be an open subset of G_α and consider $(p_\alpha \circ i)^{-1}(V_\alpha)$. We have that $p_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod_{\beta \in A \setminus \{\alpha\}} G_\beta$ is open since the projection map is continuous. By the way we defined i, we have that $i^{-1}(p_\alpha^{-1}(V_\alpha)) = i_\alpha^{-1}(V_\alpha) \times \prod_{\beta \in A \setminus \{\alpha\}} G_\beta$, and since i_α is continuous $i_\alpha^{-1}(V_\alpha)$ is open. Hence $(p_\alpha \circ i)^{-1}(V_\alpha)$ is a cylinder, which is open in the product topology, as claimed.

Exercise 2

Show that the topological group $O(p,q)$ for $p, q \geq 1$ is not compact.

Solution: We start showing that $O(1,1)$ is not compact. Recall that

$$O(1,1) = \{g \in M(2,\mathbb{R}) | g^t \mathrm{I}_{1,1} g = \mathrm{I}_{1,1}\},$$

where

$$\mathrm{I}_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we denote by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

an element of $O(1,1)$, then the defining equation gives us back the following system of equations

$$
\begin{aligned}
&c^2 - a^2 = 1, \\
&ab = cd, \\
&b^2 - d^2 = 1.
\end{aligned}
$$

Since the hyperbolic sine is bijective, there exists a unique $t \in \mathbb{R}$ such that $\sinh(t) = a$. If we substitute this value of a in the first equation, we obtain $c = \pm\cosh(t)$. Hence we must have $d = \pm\tanh(t)b$, from which it follows that $b = \pm\cosh(t)$ and $d = \sinh(t)$. In particular the set of matrices of the form

$$X_t := \begin{pmatrix} \sinh(t) & \cosh(t) \\ \cosh(t) & \sinh(t) \end{pmatrix}, \quad t \in \mathbb{R}$$

is a subgroup of $O(1,1)$. As t goes to infinity, each coordinate of the matrix X_t written above goes to infinity, hence $O(1,1)$ contains a closed subgroup which is not compact, being unbounded (here we are using the characterization of compact sets of $\mathbb{R}^n$ as closed and bounded sets). In particular neither $O(1,1)$ can be compact.

To show that $O(p,q)$ is not compact if $p, q \geq 1$, it suffices to show that it contains a subgroup isomorphic to $O(1,1)$. Recall that $O(p,q)$ is defined as it follows

$$O(p,q) = \{g \in M(p+q,\mathbb{R}) | g^t \mathrm{I}_{p,q} g = \mathrm{I}_{p,q}\},$$

where $\mathrm{I}_{p,q}$ is the diagonal matrix with the first p entries equal to 1 and the last q entries equal to -1. We can consider the map

$$\varphi : O(1,1) \to O(p,q), \quad \varphi(X) := \begin{pmatrix} \mathrm{I}_{p-1} & 0_{p-1,2} & 0_{p-1,q-1} \\ 0_{2,p-1} & X & 0_{2,q-1} \\ 0_{q-1,p-1} & 0_{q-1,2} & \mathrm{I}_{q-1} \end{pmatrix},$$

where, for every $k, l \in \mathbb{N}$, the matrices I_k and $0_{k,l}$ are the identity of order k and the zero matrix of order (k,l), respectively. This map is the required injection and we are done.

Exercise 3

Let p be a prime number. Prove that the map

$$i : \mathbb{Z} \to \mathbb{Z}_p$$

given by $i(x) = (x \mod p^n)_{n \in \mathbb{N}}$ is injective with dense image. The set of p-adic integers $\mathbb{Z}_p$ is endowed with the topology exposed during the lectures.

Solution: Recall that

$$\mathbb{Z}_p = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} | \varphi_n(x_{n+1}) = x_n\},$$

where $\varphi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ is the reduction map. If we denote an element of $\mathbb{Z}_p$ as $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, we define also the projection map $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$, given by $\pi_n(x) = x_n$.

We first prove that the map i is injective. Assume that, given $x \in \mathbb{Z}$, we have $i(x) = \mathbf{0}$, where $\mathbf{0}$ is the zero element of $\mathbb{Z}_p$. This implies that $x \equiv 0 \mod p^n$ for every $n \in \mathbb{N}$, or equivalently $p^n|x$ for every $n \in \mathbb{N}$, but this can happen only if $x = 0$, hence i is injective.

We now prove that i has dense image. To do this, recall that

$$\mathrm{Ker}(\pi_n) := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p \,|x_n = x_{n-1} = \ldots = x_1 = 0\}, \;\; n \in \mathbb{N}$$

is a fundamental system of neighborhoods of $\mathbf{0} \in \mathbb{Z}_p$. If we consider $\mathbf{x} \in \mathbb{Z}_p$, a fundamental system of neighborhoods of $\mathbf{x}$ is given by $\{\mathbf{x} + \mathrm{Ker}(\pi_n)\}_{n\in\mathbb{N}}$. We write

$$N_m(\mathbf{x}) := \mathbf{x} + \mathrm{Ker}(\pi_m) = \{(y_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p \,|x_m = y_m, \ldots, x_1 = y_1\}.$$

We want to show that, for any $m \in \mathbb{Z}$, there exists $x \in \mathbb{Z}$ such that $i(x) \in N_m(\mathbf{x})$. Note that x actually depends on m, but we do not want to overload the notation. Take now any $x \in \mathbb{Z}$ such that $x \equiv x_m \mod p^m$. In particular we have that $p^m|x - x_m$ and hence $p^k|x - x_m$ for every $k \leq m$. By the compatibility condition it follows

$$x_{m-1} \equiv x_m \mod p^{m-1} \equiv x \mod p^{m-1}.$$

Since the equation above holds for every $k \leq m$, the claim follows.

Exercise 4

Let (X, d) be a metric space. Suppose that the closed ball $B_{\leq r}(x) = \{y \in X|d(x, y) \leq r\}$ of radius r centered at x is compact, for all $r \geq 1$ and all $x \in X$. Show the set $\mathrm{Isom}(X)$ of the isometries of X is a locally compact topological group when endowed with the compact-open topology.

Solution: Denote by $G = \mathrm{Isom}(X)$. We need to show that $\mu : G \times G \to G$ given by $\mu(f, g) := f \circ g$ and the inversion $i : G \to G$ are continuous with respect to compact-open topology.

Let $f \circ g$ be in a subbasis element $S(K, U)$, where K is compact and U is open in X. We know that

$$(f \circ g)(K) \subset U$$

or equivalently

$$g(K) \subset f^{-1}(U),$$

since f is bijective. Since closed balls are compact, we can find an open neighborhood W of $g(K)$ which satisfies

$$g(K) \subset W \subset \overline{W} \subset f^{-1}(U)$$

and $\overline{W}$ is compact. Consider now $S(\overline{W}, U) \times S(K, W)$. This is an open neighborhood of (f, g) is $G \times G$. Moreover, given $(h, l) \in S(\overline{W}, U) \times S(K, W)$, it holds

$$(h \circ l)(K) \subset h(W) \subset U,$$

hence $(h \circ l) \in S(K, U)$.

We want to prove that the inversion is continuous. We are going to exploit the fact that the compact-open topology coincides with uniform convergence on compact set in our setting. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of isometries converging to an isometry f on every compact subset K of X. We need to show that $(f_n^{-1})_{n\in\mathbb{N}}$ converges to f^{-1} as well. Let K be a compact subset of X. It holds

$$\sup_K d(f_n^{-1}(x), f^{-1}(x)) = \sup_K d(f_n^{-1}(ff^{-1}(x)), f^{-1}(x)) = \sup_K d(f(f^{-1}(x)), f_n(f^{-1}(x)),$$

where the last equality comes from the fact that f_n is an isometry. Since f is an isometry as well, it is in particular a homeomorphism, hence $f^{-1}(K)$ is compact. Hence we get

$$\sup_K d(f(f^{-1}(x)), f_n(f^{-1}(x)) = \sup_{f^{-1}(K)} d(f(y), f_n(y))$$

and the last term converges to zero, as desired.

Since any isometry $f \in G$ satisfies $d(x, y) = d(f(x), f(y))$ for any $x, y \in X$, then any family $\mathcal{F} \subset G$ of isometries is automatically an equicontinuous family.

We need to show that every $f \in G$ admits a compact neighborhood. We claim that it suffices to prove this property in the particular case of f equal to the identity map id_X. Indeed if id_X admits a compact neighborhood $\mathcal{F}$, then $f(\mathcal{F})$ will be a compact neighborhood of f. Fix $x \in X$ and $r \geq 0$. We define

$$\mathcal{F}_r(x) := \{f \in G | f(B_{\leq r}(x)) \subset B_{\leq 2r}(x)\}.$$

The set defined above contains the identity and it is equicontinuous for what we said before. Let $y \in B_{\leq r}(x)$. Since $d(f(y), x) \leq 2r$ for every $f \in \mathcal{F}_r(x)$, the closed set $\overline{\{f(y)|f \in \mathcal{F}_r(x)\}}$ is compact, being a closed subset of a compact set $B_{\leq 2r}(x)$ (recall that this set is compact by hypothesis). Since f is an isometry then for every $z \in X$ we have that $\overline{\{f(z)|f \in \mathcal{F}_r(x)\}}$ lies in a closed ball. Indeed if $y \in B_{\leq r}(x)$ it holds

$$d(f(z), x) \leq d(f(z), f(y)) + d(f(y), x) \leq d(z, y) + r,$$

Hence $\overline{\{f(z)|f \in \mathcal{F}_r(x)\}}$ is compact for every $z \in X$ and this implies that $\mathcal{F}_r(x)$ is pointwise bounded. We desume that $\mathcal{F}_r(x)$ is compact by the Ascoli-Arzela theorem.

Exercise 5

Let G be a connected topological group and let V be a connected neighborhood of the neutral element. Show that there exists a neighborhood W such that $W^2 \subset V$ and $W = W^{-1}$.

Solution: We know that the multiplication $\mu : G \times G \to G$ is continuous. In particular $\mu^{-1}(V)$ is an open set containing $(e, e) \in G \times G$. By the definition of product topology, there must exist $V_1, V_2 \subset G$ open subsets containing e such that $V_1 \times V_2 \subset \mu^{-1}(V)$. Define $U = V_1 \cap V_2$ and $W = U \cap U^{-1}$. We claim that W is the neighborhood we were looking for. Obviously, by definition, we have $W = W^{-1}$. We need to show that $W^2 \subset V$. Let $x, y \in W$ and consider their product $xy = \mu(x, y)$. In particular $x, y \in V_1 \cap V_2$ and hence $(x, y) \in V_1 \times V_2$ from which we deduce $xy \in V$, as claimed.