Exercise Sheet 2

Exercise 1

Let G be a locally compact Hausdorff group. Show that

1. The modular function $\Delta_G : G \to \mathbb{R}_{>0}$ is continuous.
2. $\int_G f(x^{-1})\Delta_G(x^{-1})d\mu(x) = \int_G f(x)d\mu(x)$.

Solution: 1. To prove the continuity of the modular function Δ_G, we need first to prove the following fact. If $f \in C_{00}(G)$, for any $\varepsilon > 0$, there exists a suitable neighborhood V of e such that

$$|f(xy) - f(x)| < \varepsilon$$

for every $y \in V$ and every $x \in G$.

Set $K = \mathrm{supp}(f)$. For any $x \in K$ there exists V_x neighborhood of e such that

$$|f(xy) - f(x)| < \varepsilon/2$$

for every $y \in V_x$. Without loss of generality we can assume $V_x^2 \subset V_x$ and $V_x = V_x^{-1}$. The family $\mathfrak{U} := \{xV_x\}_{x\in K}$ is an open covering of K and since the latter is compact, we can extract a finite subcovering of $\mathfrak{U}$. Let $\{x_0V_0, \ldots, x_nV_n\}$ be this subcovering, where we set $V_i = V_{x_i}$. Define $U := \bigcap_{i=0}^n V_i$. We claim that U is the desired neighborhood, that is

$$|f(xy) - f(x)| < \varepsilon$$

for every $x \in G$ and every $y \in U$. We are going to show the statement above only for $x \in K$ since the argument can be suitably adapted to the case $x \notin K$. Assume $x \in K$. There must exists $i \in \{0, \ldots, n\}$ such that $x \in x_iV_i$, or equivalently $x_i^{-1}x \in V_i$. For any $y \in U$ we can write

$$xy = x_ix_i^{-1}xy$$

and $x_i^{-1}xy \in V_i$ by the way we constructed V_i. Hence it holds

$$|f(xy) - f(x_i)| < \varepsilon/2.$$

Similarly we will have

$$|f(x) - f(x_i)| < \varepsilon/2,$$

thus we deduce that

$$|f(xy) - f(x)| < \varepsilon,$$

as desired. We are ready now to prove continuity of Δ_G. Since Δ_G is a morphism of topological groups, we are going to prove continuity at $e \in G$. For every $\varepsilon > 0$ we want to find a neighborhood U of e such that

$$|\Delta_G(g) - 1| < \varepsilon$$

for every $g \in U$. Let $f \in C_{00}(G)$. Recall that it holds

$$\int_G f(xg^{-1})d\mu(x) = \Delta_G(g) \int_G f(x)d\mu(x).$$

Hence we can write

$$|\Delta_G(g)-1| = |\int_G (f(xg^{-1})-f(x))d\mu(x)|.$$

Let C be a compact neighborhood of e (it exists because G is locally compact). As a consequence of what we showed before, for any ε there exists a neighborhood U of e such that

$$|f(xg^{-1})-f(x)| < \varepsilon/\mu(CK)$$

for any $g \in U$ and any $x \in G$ (we can suppose $U = U^{-1}$). Notice that CK is compact by the continuity of multiplication in G. Additionally we can assume that $U \subset C$.

It holds

$$|\int_G (f(xg^{-1})-f(x))d\mu(x)| \leq \int_G |f(xg^{-1})-f(x)|d\mu(x) \leq \mu(CK)\varepsilon/\mu(CK) = \varepsilon$$

for any $g \in U$ and the statement follows.

2. It easy to see that that the function

$$\overline{m} : C_{00}(G) \to \mathbb{R},\ \ \overline{m}(f) := \int_G f(x^{-1})\Delta(x^{-1})d\mu(x)$$

defines another left Haar measure on G, hence by uniqueness there must exists a constant $c \in (0,\infty)$ such that

$$\int_G f(x^{-1})\Delta_G(x^{-1})d\mu(x) = c\int_G f(x)d\mu(x).$$

Since we showed that Δ_G is continuous (and the inversion in G is continuous), if we fix $\varepsilon > 0$, there must exist a suitable neighborhood V of e such that

$$|\Delta_G(x^{-1})-1| < \varepsilon$$

for every $x \in V$. We choose now a positive function $h \in C_{00}(G)$ such that $h(x) = h(x^{-1})$ and $\mathrm{supp}(h) \subset V$. We have

$$\int_G h(x^{-1})\Delta_G(x^{-1})d\mu(x) = \int_G h(x)\Delta_G(x^{-1})d\mu(x),$$

hence it follows

$$|c-1|\int_G h(x)d\mu(x) = |\int_G (\Delta_G(x^{-1})-1)h(x)d\mu(x)| \leq \int_G |\Delta_G(x^{-1})-1|h(x)d\mu(x) \leq \varepsilon\int_G h(x)d\mu(x)$$

and since ε is arbitrary, we are done.

Exercise 2

Show that the modular function associated to the group

$$P = \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x > 0, y \in \mathbb{R}\}$$

is given by

$$\Delta_P \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-2}.$$

Solution: It is easy to check that the linear application

$$m : C_{00}(P) \to \mathbb{R},\ \ m(f) := \int_{\mathbb{R}} \int_0^\infty f \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} x^{-2} dx dy$$

defines a left Haar measure for P. To compute the modular function Δ_P we pick an element

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$$

and we consider

$$\int_{\mathbb{R}} \int_0^\infty f \left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \right) x^{-2} dx dy = \int_{\mathbb{R}} \int_0^\infty f \begin{pmatrix} a^{-1}x & -xb + ya \\ 0 & ax^{-1} \end{pmatrix} x^{-2} dx dy,$$

where we used that left invariance property of the Haar measure.

Define now

$$X = a^{-1}x, \quad Y = -bx + ay.$$

With this change of coordinates, we can express the differentials dx and dy as it follows

$$dx = a^{-1}dX, \quad dy = adY + bdX.$$

By substituting in the integral we get

$$\int_{\mathbb{R}} \int_0^\infty f \begin{pmatrix} a^{-1}x & -xb + ya \\ 0 & ax^{-1} \end{pmatrix} x^{-2} dx dy = a^{-2} \int_{\mathbb{R}} \int_0^\infty f \begin{pmatrix} X & Y \\ 0 & X^{-1} \end{pmatrix} X^{-2} dX dY$$

and hence $\Delta_G \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-2}$.

Exercise 3

Let G be a topological group and let H be a subgroup of G. Show that the map

$$G \times G/H \to G/H,\ \ (g, xH) \mapsto gxH,$$

is continuous.

Solution: Let us denote by μ the multiplication in G, by $\pi : G \to G/H$ the quotient map and by $\theta : G \times G/H \to G/H$ the action map. It is clear that we have the following commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\times\pi"] & G \arrow[d, "\pi"] \\ G \times G/H \arrow[r, "\theta"] & G/H \end{tikzcd}$$

Let U be an open set in G/H. We are going to show that every point $(g, xH) \in \theta^{-1}(U)$ admits an open neighborhood completely contained in $\theta^{-1}(U)$. Consider now $\mu^{-1}(\pi^{-1}(U))$ in $G \times G$. Since both μ and π are continuous, $\mu^{-1}(\pi^{-1}(U))$ is open. The point (g, x) lies in $\mu^{-1}(\pi^{-1}(U))$ by commutativity of the diagram, hence there exists V open neighborhood of g and W open neighborhood of x such that $V \times W \subset \mu^{-1}(\pi^{-1}(U))$.

Recall now that the projection map $\pi : G \to G/H$ is open. This implies that $V \times \pi(W)$ is open in $G \times G/H$ and contains (g, xH). Again by commutativity of the diagram above

$$\theta(V \times \pi(W)) = \pi(\mu(V \times W)) \subset U$$

and we are done.

Exercise 4

Let G be a locally compact Hausdorff group which is separable. Assume that $G \times X \to X$ is a transitive continuous action on a locally compact Hausdorff space X. Show that the map

$$G/\mathrm{Stab}_G(x_0) \to X, \;\; g\mathrm{Stab}_G(x_0) \mapsto gx_0$$

is a homeomorphism.

Solution: Let us denote by $\pi : G \to G/\mathrm{Stab}_G(x_0)$ the projection map and by $\mathrm{e}_{x_0} : G/\mathrm{Stab}_G(x_0) \to X$ the evaluation map at x_0. It should be clear that we have the following commutative diagram

$$\begin{tikzcd} G \arrow[d, "\pi"] \arrow[dr] & \\ G/\mathrm{Stab}_G(x_0) \arrow[r, "\mathrm{e}_{x_0}"'] & X. \end{tikzcd}$$

The injectivity of the map e_{x_0} is obvious. The surjectivity comes from the fact that the action of G on X is transitive. The continuity is a direct consequence of the properties of quotient topology.

We are left to show that the map e_{x_0} is open. In order to do this, let U be an open subset of G. We need to show that Ux_0 is open in X, that is for any $g \in U$ the point gx_0 admits an open neighborhood completely contained in Ux_0 (equivalently gx_0 is an interior point of Ux_0). Let V be a compact neighborhood of the neutral element such that $V = V^{-1}$ and that $gV^2 \subset U$ (it exists by what we have shown in the previous exercise sheet).

Since G is separable, there exists a countable subset $D = \{g_n\}_{n \in \mathbb{N}}$ which is dense. In particular we have $G = \bigcup_{n=1}^{\infty} g_n V$. This implies that $X = \bigcup_{n=1}^{\infty} g_n V x_0$. The continuity of the action implies that each set $g_n V x_0$ is compact and hence closed by the Hausdorff property of X. In this way, we have written a locally compact space X as a countable union of closed spaces. By the Baire Category theorem at least one of these set must have non-empty interior. Hence there exists $m \in \mathbb{N}$ such that $g_m V x_0$ contains an interior point, or equivalently $V x_0$ contains an interior point (since translations by element of G are homeomorphism of X). This means there exists $h \in V$ such that hx_0 admits an open neighborhood completely contained in Vx_0. Thus $gx_0 = gh^{-1}hx_0$ is an interior point of Ux_0 and we are done.