Exercise Sheet 3

Exercise 1

Let G be a locally compact Hausdorff group with left Haar measure μ. Let φ_1, φ_2 be two measurable functions on G. We can define the convolution $\varphi_1 * \varphi_2$ as it follows

$$\varphi_1 * \varphi_2(g) := \int_G \varphi_1(h^{-1}g)\varphi_2(h)d\mu(h),$$

whenever the integral above is finite for almost every $g \in G$.

1. Assume $\varphi_1, \varphi_2 \in C_{00}(G)$. Prove that $\varphi_1 * \varphi_2$ is continuous and it has compact support.
2. Let $\varphi_1, \varphi_2 \in L^1(G, \mu)$. Prove that $\varphi_1 * \varphi_2$ exists almost everywhere, it is an element of $L^1(G, \mu)$ and it holds

   $$||\varphi_1 * \varphi_2||_1 \leq ||\varphi_1||_1 ||\varphi_2||_1,$$

   where $|| \cdot ||_1$ denotes the L^1-norm.
3. Show that if $\varphi \in L^1(G, \mu)$ and $f \in L^\infty(G, \mu)$ then the convolution $\varphi * f$ is almost everywhere equal to a continuous function.

Exercise 2

Let $\Lambda < \mathbb{R}^2$ be a lattice such that $\mathrm{Vol}(\mathbb{R}^2 / \Lambda) = 1$. Prove that

$$\lim_{R \to \infty} \frac{|\Lambda_{\mathrm{prim}} \cap B(0, R)|}{\mathrm{Vol}(B(0, R))} = \frac{1}{\zeta(2)},$$

where $| \cdot |$ is the cardinality of the set and $B(0, R)$ is the ball around 0 of radius R. Recall that Λ_{prim} is the set of primitive elements of Λ and $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.