Exercise Sheet 4

Exercise 1

Let G, H be two Lie groups and let $\varphi : G \to H$ be a smooth homomorphism. Show that φ has constant rank.

Exercise 2

Let M be a smooth manifold and let $p \in M$ a point. Denote by $C^\infty(p)$ the ring of germs of functions which are smooth at p.

1. Show that
$$\mathfrak{m}_p := \{f \in C^\infty(p) : f(p) = 0\}$$
is a maximal ideal of $C^\infty(p)$.

2. Let $\mathfrak{m}_p^2$ the ideal generated by all the products of the form $f \cdot g$, where $f, g \in \mathfrak{m}_p$. Show that the tangent space T_pM is canonically isomorphic to the dual space $(\mathfrak{m}_p \,/\, \mathfrak{m}_p^2)^*$ as $\mathbb{R}$-vector space.

Exercise 3

Show that the cross product $\wedge$ on $\mathbb{R}^3$ gives it a structure of Lie algebra.

Exercise 4

Show that it holds
$$(D_{\mathrm{Id}} \det)(X) = \mathrm{tr}(X)$$
for every $X \in M(n, \mathbb{R})$.

Exercise 5

Compute explicitly the Lie algebra of the group $O(p, q)$ for every p, q.

Exercise 6

Let G, H be Lie groups with associated Lie algebras $\mathfrak{g}, \mathfrak{h}$. Verify that the Lie algebra of the product $G \times H$ is $\mathfrak{g} \times \mathfrak{h}$ where the bracket on the latter is given by
$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2], [Y_1, Y_2]),$$
where $X_1, X_2 \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{h}$.

Exercise 7

Show that $Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$ and the group $U(n)$ are isomorphic as Lie groups.