Exercise Sheet 4

Exercise 1

Let G, H be two Lie groups and let $\varphi : G \to H$ be a smooth homomorphism. Show that φ has constant rank.

Solution: We need to show that for every $g \in G$ the rank of the linear map

$$D_g\varphi : T_gG \to T_{\varphi(g)}H$$

is constant, that is it does not depend on g. Since φ is a homomorphism, it should be clear that we have the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "L_g"] & H \arrow[d, "L_{\varphi(g)}"] \\ G \arrow[r, "\varphi"] & H, \end{tikzcd}$$

where L_g and $L_{\varphi(g)}$ are the maps given by left translation by g and $\varphi(g)$, respectively. By the structure of Lie groups and by the smoothness of the homomorphism φ, the diagram above induces the following commutative diagram

$$\begin{tikzcd} T_eG \arrow[r, "D_e\varphi"] \arrow[d, "D_eL_g"] & T_eH \arrow[d, "D_eL_{\varphi(g)}"] \\ T_gG \arrow[r, "D_g\varphi"] & T_{\varphi(g)}H. \end{tikzcd}$$

Since both D_eL_g and $D_eL_{\varphi(g)}$ are isomorphisms, it is clear that

$$\mathrm{rank} D_g\varphi = \mathrm{rank} D_e\varphi$$

for every $g \in G$ and we are done.

Exercise 2

Let M be a smooth manifold and let $p \in M$ a point. Denote by $C^\infty(p)$ the ring of germs of functions which are smooth at p.

1. Show that
$$\mathfrak{m}_p := \{f \in C^\infty(p) : f(p) = 0\}$$
is a maximal ideal of $C^\infty(p)$.

2. Let $\mathfrak{m}_p^2$ the ideal generated by all the products of the form $f \cdot g$, where $f, g \in \mathfrak{m}_p$. Show that the tangent space T_pM is canonically isomorphic to the dual space $(\mathfrak{m}_p / \mathfrak{m}_p^2)^*$ as $\mathbb{R}$-vector space.

Solution: To show that $\mathfrak{m}_p$ is a maximal ideal of $C^\infty(p)$ we are going to show that the quotient $C^\infty(p)/\mathfrak{m}_p$ is a field isomorphic to $\mathbb{R}$ (recall that $\mathfrak{m}$ is a maximal ideal of a ring R iff $R/\mathfrak{m}$ is a field). We define the evaluation map at p as it follows

$$\mathrm{ev}_p : C^\infty(p) \to \mathbb{R}, \ \ \mathrm{ev}_p[f] := f(p),$$

where $[f]$ denotes the class of the function f in $C^\infty(p)$. First of all we notice that the evaluation map is well-defined. Moreover it is clear that it is a ring homomorphism which is surjective and whose kernel is equal to

$$\ker(\mathrm{ev}_p) = \mathfrak{m}_p\,.$$

Hence we have the following isomorphism of rings

$$C^\infty(p)/\,\mathfrak{m}_p = C^\infty(p)/\ker(\mathrm{ev}_p) \cong \mathbb{R},$$

and since $\mathbb{R}$ is a field, the same must hold for $C^\infty(p)/\,\mathfrak{m}_p$, as desired.

Solution: In this section, to avoid heavy notation, we are going to drop brackets to denote the class of a function in $C^\infty(p)$.

Recall that among all the possible way to define it, the tangent space T_pM at p can be seen as the $\mathbb{R}$-vector space of derivations of $C^\infty(p)$. More precisely any element $X \in T_pM$ can be thought of as a $\mathbb{R}$-linear map

$$X : C^\infty(p) \to \mathbb{R},$$

which satisfies the following property

$$X(fg) = f(p)X(f) + g(p)X(f).$$

We briefly recall some facts. Let us fix a coordinate chart $(U, (x_1, \ldots, x_m))$ around p, such that p has coordinates $x_1 = \ldots = x_m = 0$. A possible basis for the tangent space at p is given by

$$\mathcal{B} = \{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_m}|_p\},$$

that means that every tangent vector X can be written as

$$X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}|_p,$$

with $a_i \in \mathbb{R}$ for $i = 1, \ldots, m$. In particular the action of X on a function $f \in C^\infty(p)$ is given by

$$X(f) := \sum_{i=1}^m a_i \frac{\partial f}{\partial x_i}(p).$$

By the discussion above, we immediately see that X is an element of the dual $C^\infty(p)^*$. We define a map

$$\mathrm{res} : T_pM \to \mathfrak{m}_p^*,\ \ \mathrm{res}(X) := X|_{\mathfrak{m}_p},$$

where $X|_{\mathfrak{m}_p}$ is the restriction to $\mathfrak{m}_p$ of the derivation X. This means that X has to be interpreted as a linear functional $X : \mathfrak{m}_p \to \mathbb{R}$. Since X is a derivation it should be clear that X is identically zero on $\mathfrak{m}_p^2$. Indeed, let $f, g \in \mathfrak{m}_p$. It holds

$$X(fg) = f(p)X(g) + g(p)X(f) = 0,$$

since both f and g satisfy $f(p) = g(p) = 0$. By this condition, we get a well-defined map

$$\Phi : T_pM \to (\mathfrak{m}_p\,/\,\mathfrak{m}_p^2)^*$$

induced by the restriction.

We first prove injectivity of Φ. Assume that X induces the zero functional on $\mathfrak{m}_p$. Since X has the form

$$X := \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}|_p,$$

for suitable values of $a_i \in \mathbb{R}$, it must hold

$$X(f) := \sum_{i=1}^{m} a_i \frac{\partial f}{\partial x_i}(p) = 0,$$

for every $f \in C^\infty(p)$ with $f(p) = 0$. This implies immediately that $a_i = 0$ for all $i = 1, \ldots, m$, hence $X = 0$ and Φ is injective.

We now prove surjectivity. Let $\varphi : \mathfrak{m}_p \to \mathbb{R}$ be a linear function which vanishes on $\mathfrak{m}_p^2$. Since the coordinate function $x_i : U \to \mathbb{R}$ determines a well-defined element of $\mathfrak{m}_p$, we can define

$$X := \sum_{i=1}^{m} \varphi(x_i) \frac{\partial}{\partial x_i}|_p.$$

By a standard result in differential geometry any function $f \in \mathfrak{m}_p$ can be written in coordinates $(x_1, \ldots, x_n)$ as it follows

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p) x_i + \sum_{i=1}^{m} g_i(x_1, \ldots, x_m) x_i,$$

where each $g_i \in \mathfrak{m}_p$. This implies that $\varphi(f)$ is completely determined by the values assumed by φ on the functions x_i, hence $X = \varphi$ and we are done.

Exercise 3

Show that the cross product $\wedge$ on $\mathbb{R}^3$ gives it a structure of Lie algebra.

Solution: It is a simple verification and we omit it.

Exercise 4

Show that it holds

$$(D_{\mathrm{Id}} \det)(X) = \mathrm{tr}(X)$$

for every $X \in M(n, \mathbb{R})$.

Solution: Let $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ be the determinant map. We know that this is a differentiable homomorphism (the determinant is a polynomial in terms of the coordinates of the matrix). Let $X \in M(n, \mathbb{R})$ be any matrix. To compute the differential of the determinant at the identity Id we have to compute

$$(D_{\mathrm{Id}} \det)(X) = \frac{d}{dt}|_{t=0} \det(\mathrm{Id} + tX).$$

Indeed $c(t) := \mathrm{Id} + tX$ is a smooth curve passing through Id, completely contained in $GL(n, \mathbb{R})$ for small values of t and with derivative at zero equal to $\dot{c}(0) = X$ (recall the computation of the differential of a smooth map in terms of smooth curves). We are going to exploit the Leibniz formula for the determinant. If we denote by $\mathfrak{S}_n$ the symmetric group on n elements and by $\varepsilon(\sigma)$ the sign of a permutation $\sigma \in \mathfrak{S}_n$, we have

$$\begin{aligned}\frac{d}{dt}|_{t=0} \det(\mathrm{Id} + tX) =& \frac{d}{dt}|_{t=0} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\mathrm{Id}+tX)_{\sigma(1)1}\dots(\mathrm{Id}+tX)_{\sigma(n)n} = \\ =& \frac{d}{dt}|_{t=0} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\delta_{\sigma(1)1}+tX_{\sigma(1)1})\dots(\delta_{\sigma(n)n}+tX_{\sigma(n)n}),\end{aligned}$$

where δ_{ij} is the Kronecker symbol. It shoud be clear that we can write

$$\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\delta_{\sigma(1)1}+tX_{\sigma(1)1})\dots(\delta_{\sigma(n)n}+tX_{\sigma(n)n}) = a_0 + a_1 t + t^2 q(t),$$

where $a_0, a_1 \in \mathbb{R}$ and $q(t) \in \mathbb{R}[t]$. In particular, it holds

$$\frac{d}{dt}|_{t=0} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\delta_{\sigma(1)1}+tX_{\sigma(1)1})\dots(\delta_{\sigma(n)n}+tX_{\sigma(n)n}) = \frac{d}{dt}|_{t=0}(a_0+a_1t+t^2q(t)) = a_1.$$

Everything boils down to compute the coefficient a_1, that means the coefficient in degree 1 of the polynomial $\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\delta_{\sigma(1)1}+tX_{\sigma(1)1})\dots(\delta_{\sigma(n)n}+tX_{\sigma(n)n})$. It should be clear that we have

$$a_1 = \sum_{\sigma\in\mathfrak{S}_n} \sum_{i=1}^n \varepsilon(\sigma)\delta_{\sigma(1)1}\dots X_{\sigma(i)i}\dots\delta_{\sigma(n)n},$$

that means the in all the possible products above the only coefficient of the matrix X appears with indices equal to $(\sigma(i), i)$. The only non-vanishing term is given when σ is the identity, otherwise at least one of the Kronecker symbol is equal to zero. Hence we have

$$a_1 = \sum_{\sigma\in\mathfrak{S}_n} \sum_{i=1}^n \varepsilon(\sigma)\delta_{\sigma(1)1}\dots X_{\sigma(i)i}\dots\delta_{\sigma(n)n} = \sum_{i=1}^n X_{ii} = \mathrm{tr}(X).$$

Then we obtained

$$(D_{\mathrm{Id}} \det)(X) = a_1 = \mathrm{tr}(X),$$

and we are done.

Exercise 5

Compute explicitly the Lie algebra of the group $O(p,q)$ for every p,q.

Solution: Denote by $n := p+q$. Recall that the definition of the group $O(p,q)$ is given by

$$O(p,q) = \{X \in GL(n,\mathbb{R}) | {}^tX I_{p,q} X = I_{p,q}\}.$$

To compute the Lie algebra associated to $O(p,q)$ we are going to realize this group as a fiber of a suitable constant rank map. We define

$$F : GL(n,\mathbb{R}) \to GL(n,\mathbb{R}), \quad F(X) := {}^tXI_{p,q}X.$$

Clearly F is smooth since it can be expressed as a polynomial functions of the coordinates of the matrix X. Additionally, by definition we have $F^{-1}(I_{p,q}) = O(p,q)$. We are going to prove that the map F has constant rank. Let X be any element in $GL(n,\mathbb{R})$ and let Y be any tangent vector at X (that means $Y \in M(n,\mathbb{R})$). Using the usual definition of the differential in terms of smooth curves we have

$$\begin{aligned}(D_XF)(Y) =&\frac{d}{ds}|_{s=0}F(X+sY) = \frac{d}{ds}|_{s=0}({}^t(X+sY)I_{p,q}(X+sY)) = \\ =&\frac{d}{ds}|_{s=0}({}^tXI_{p,q}X + s({}^tXI_{p,q}Y + {}^tYI_{p,q}X) + s^2({}^tYI_{p,q}Y)) = \\ =&({}^tXI_{p,q}Y + {}^tYI_{p,q}X) = {}^tX(I_{p,q}YX^{-1} + {}^t(X^{-1}){}^tYI_{p,q})X = \\ =&{}^tXD_{\mathrm{Id}}(YX^{-1})X.\end{aligned}$$

From the equation above we deduce that the rank of F is constant and the Lie algebra of $O(p,q)$ is given by

$$\mathfrak{o}(p,q) = \mathrm{Lie}(O(p,q)) = \ker(D_{\mathrm{Id}}F) = \{X \in M(n,\mathbb{R}) | {}^tXI_{p,q} + I_{p,q}X = 0\}.$$

Exercise 6

Let G,H be Lie groups with associated Lie algebras $\mathfrak{g},\mathfrak{h}$. Verify that the Lie algebra of the product $G \times H$ is $\mathfrak{g}\times\mathfrak{h}$ where the bracket on the latter is given by

$$[(X_1,Y_1),(X_2,Y_2)] := ([X_1,X_2],[Y_1,Y_2]),$$

where $X_1,X_2 \in \mathfrak{g}$ and $Y_1,Y_2 \in \mathfrak{h}$.

Solution: We are going to denote by $\mathfrak{X}(M) = \mathrm{Vect}^\infty(M)$ the set of vector fields over a generic manifold M.

We are going to denote by

$$i_G : G \to G\times H, \quad i_G(g) := (g,e)$$

and similarly

$$i_H : H \to G\times H, \quad i_H(h) := (e,h).$$

In the same way, the differential of both maps induces inclusions

$$D_ei_G : T_eG \to T_eG \times T_eH, \quad D_ei_G(u) := (u,0)$$

and

$$D_ei_H : T_eH \to T_eG\times T_eH, \quad D_ei_G(v) := (0,v).$$

Recall that $T_eG\times T_eH$ is canonically isomorphic to $T_eG\oplus T_eH$ as $\mathbb{R}$-vector spaces via the map which sends (u,v) to $u+v$, for every $u\in T_eG$ and every $v\in T_eH$. (In this way we get D_ei_G is simply the inclusion of T_eG into $T_eG\oplus T_eH$ and the same for D_ei_H). This means that every element w in $T_{(e,e)}(G\times H)$ can be written uniquely as $w = u + v$, where $u \in T_eG$ and $v \in T_eH$, or equivalently we can identify $T_{(e,e)}(G \times H)$ with $T_eG \oplus T_eH$.

From the lecture, we know that there is a bijection between left-invariant vector fields on G (resp. H) and vectors of the tangent space T_eG (resp. T_eH) and the isomorphism is given by

$$L_G : T_eG \to \mathfrak{X}(G)^G, \quad L_G(u) := u^L$$

where the vector field u^L is defined at the point $g \in G$ as $u^L_g := D_eL_g(u)$.

It should be clear that we have the following commutative diagram

$$\begin{tikzcd} T_eG \oplus T_eH \arrow[r, "\cong"] \arrow[d, "L_G \oplus L_H"] & T_{(e,e)}(G \times H) \arrow[d, "L_{G \times H}"] \\ \mathfrak{X}(G)^G \oplus \mathfrak{X}(H)^H \arrow[r, "\cong"] & \mathfrak{X}(G \times H)^{G \times H}. \end{tikzcd}$$

The diagram above is telling us that every $(G \times H)$-left-invariant vector field $Z = w^L$, where $w \in T_{(e,e)}(G \times H)$, can be uniquely written as $Z = X + Y$, where $X = u^L$ (resp. $Y = v^L$) where $u \in T_eG$ (resp. $v \in T_eH$). Here the left-invariance property has to be understood in $G \times H$ (that means that both u^L and v^L are $G \times H$ left-invariant).

Take now $Z_1, Z_2 \in \mathfrak{X}(G \times H)^{G \times H}$ of the form $Z_i = w_i^L$, where $w_i \in T_{(e,e)}(G \times H)$ for $i = 1, 2$. By what we have said so far there exist unique $u_i \in T_eG$ and $v_i \in T_eH$ such that $w_i^L = u_i^L + v_i^L$, for $i = 1, 2$. It holds

$$\begin{aligned} [Z_1, Z_2] =& [w_1^L, w_2^L] = [u_1^L + v_1^L, u_2^L + v_2^L] = \\ =& [u_1^L, u_2^L] + [u_1^L, v_2^L] + [v_1^L, u_2^L] + [v_1^L, v_2^L]. \end{aligned}$$

It is immediate to verify that for every $[u^L, v^L] = 0$ for any $u \in T_eG$ and $v \in T_eH$, hence we get

$$[w_1^L, w_2^L] = [u_1^L, u_2^L] + [v_1^L, v_2^L],$$

which is exactly the Lie algebra structure given on the product, and we are done.

Exercise 7

Show that $Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$ and the group $U(n)$ are isomorphic as Lie groups.

Solution: We fix the following notation

$$J_{2n} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

where 0_n and I_n denote the zero matrix and the identity matrix of order n, respectively. Recall that there exists a suitable basis such that we can write

$$Sp(2n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) | {}^t g J_{2n} g = J_{2n}\}.$$

We are going to write g as a block matrix as it follows

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in M(n, \mathbb{R})$. The condition of being a real symplectic matrix gives us back the following equations

$$^tCA = {}^tAC, \ \ ^tDB = {}^tBD, \ \ ^tAD - {}^tCB = I_n. \tag{1}$$

If now require that g is orthogonal, that means $g \in O(2n, \mathbb{R})$, we get another list of equations

$$^tAA + {}^tCC = I_n, \ \ ^tBB + {}^tDD = I_n, \ \ ^tAB + {}^tCD = 0_n. \tag{2}$$

We set now

$$X := A + iC, \quad Y := B + iD.$$

It is easy to see that both X and Y are unitary. Let us check it for X

$$^t\bar{X}X = {}^t(A - iC)(A + iC) = ({}^tAA + {}^tCC) + i({}^tAC - {}^tCA) = I_n,$$

by both Equations (1) and Equations (2). In the same we get that Y is unitary. In this way we constructed $X, Y \in U(n)$. We are going to prove now that $A = D$ and $C = -B$. Indeed it holds

$$^t\bar{X}Y = {}^t(A - iC)(B + iD) = ({}^tAB + {}^tCD) + i({}^tAD - {}^tCB) = iI_n,$$

again by both Equations (1) and Equations (2). In particular this means that we have $^t\bar{X} = iY^{-1}$ and $^t\bar{X} = X^{-1}$ at the same time, and by uniquess it must hold $X = iY$ which means exactly $A = D$ and $C = -B$. So we get that

$$g \in Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) \Leftrightarrow g = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where $A, B \in M(n, \mathbb{R})$ such that $X := A + iB \in U(n)$. Hence we can define

$$\Phi : Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) \to U(n), \quad \Phi \begin{pmatrix} A & -B \\ B & A \end{pmatrix} := A + iB,$$

and one can show that this is the desired isomorphism of Lie groups.