# Exercise Sheet 5

## Exercise 1

Show that $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$. (*Hint:* If $\dim V = 4$ then $\dim \Lambda^2 V = 6$.)

**Solution:** Consider the standard action of an element $g \in \mathrm{SL}(4,\mathbb{C})$ on the vector space $\mathbb{C}^4$, that means

$$g.v := gv, \quad v \in \mathbb{C}^4,$$

where $gv$ is the standard multiplication rows-by-columns. This action determines an action on the space $\mathbb{C}^4 \otimes \mathbb{C}^4$ and hence on the space $\Lambda^2\,\mathbb{C}^4$ given by

$$g.(u \wedge v) := gu \wedge gv, \quad u, v \in \mathbb{C}^4\,.$$

In this way we obtain a morphism $\mathrm{SL}(4,\mathbb{C}) \to \mathrm{SL}(6,\mathbb{C})$. We need to show that its image actually is contained in $\mathrm{SO}(6,\mathbb{C})$.

The key point now is that $\Lambda^4\,\mathbb{C}^4 \cong \mathbb{C}$ and if we fix the canonical basis $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ a generator is given by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. This allows us to define a symmetric bilinear form $B : \Lambda^2\,\mathbb{C}^4 \times \Lambda^2\,\mathbb{C}^4 \to \mathbb{C}$ as it follows. Consider $u_1 \wedge v_1, u_2 \wedge v_2 \in \Lambda^2\,\mathbb{C}^4$. Since $\Lambda^4\,\mathbb{C}^4$ is generated by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$, we can define $B(u_1 \wedge v_1, u_2 \wedge v_2)$ to be the unique scalar for which it holds

$$u_1 \wedge v_1 \wedge u_2 \wedge v_2 = B(u_1 \wedge v_1, u_2 \wedge v_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

The function $B$ defined above is a symmetric bilinear form which is non-degenerate (you can express the associated matrix in the basis $\Lambda^2\,\mathcal{E} = \{e_i \wedge e_j\}_{i<j}$ and check that the determinant is different from zero). In addition the $\mathrm{SL}(4,\mathbb{C})$-action on $\Lambda^2\,\mathbb{C}^4$ preserves $B$ (you can check it on the elements on the basis since $B$ is bilinear). Thus the representation $\mathrm{SL}(4,\mathbb{C}) \to \mathrm{SL}(6,\mathbb{C})$ has image contained in $\mathrm{SO}(6,\mathbb{C})$, as desired. This is a smooth homomorphism whose kernel is equal to $\{\mathrm{Id}, -\mathrm{Id}\}$, hence it induces an isomorphism between the associated Lie algebras, as desired.

## Exercise 2

Show that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

**Solution:** Using the definition it is easy to verify that $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Sp}(2,\mathbb{C})$ (see Exercise 4 below). This means that any $g \in \mathrm{SL}(2,\mathbb{C})$ preserves the standard symplectic form given by

$$\omega(u,v) := {}^t u J_2 v, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for every $u, v \in \mathbb{C}^2$.

By using the symplectic form $\omega$ we can define the following function

$$B : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}, \quad B(u_1 \otimes v_1, u_2 \otimes v_2) = \omega(u_1, u_2)\omega(v_1, v_2).$$

The function $B$ is a symmetric bilinear form which is non-degenerate. If we now consider the action of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by $(g,h)(u \otimes v) := (gu \otimes hv)$ for every $g, h \in \mathrm{SL}(2,\mathbb{C}), u, v \in \mathbb{C}^2$, we get that this action preserves $B$. Hence we get a smooth homomorphism $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C})$ whose kernel is $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$. The induced homorphism on the associated Lie algebras is the desired isomorphism.

## Exercise 3

Show that $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$.

**Solution:** Recall that the Lie algebra

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in M(2,\mathbb{C}) |\, \mathrm{tr}(X) = 0\}$$

admits the following natural basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the same way the Lie algebra

$$\mathfrak{so}(3,\mathbb{C}) = \{X \in M(3,\mathbb{C}) |\, {}^t X + X = 0\}$$

has a natural basis given by $h_{ij} = E_{ij} - E_{ji}$ for $i,j = 1,2,3$ and $i < j$. Here $E_{ij}$ denotes the matrix with 1 in the only entry of indices $(i,j)$ and equal to zero elsewhere.

Define a map $\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{C})$ in the following way

$$\varphi(H) = -2ih_{13}, \;\; \varphi(E) = ih_{12} + h_{23}, \;\; \varphi(F) = -ih_{12} + h_{23}. \tag{1}$$

It easy to verify that the map $\varphi$ gives us the desired isomomorphism.

## Exercise 4

Show that $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$.

**Solution:** It follows immediately by the definition that $\mathrm{Sp}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$ and from this it follows the isomorphism between the associated Lie algebras (they are the same).

We move now to $\mathrm{Sp}(4,\mathbb{C})$. Consider the standard action of $\mathrm{Sp}(4,\mathbb{C})$ on $\Lambda^2\,\mathbb{C}^4$ (the same defined in Exercise 1) and the same bilinear form $B$. Since $\mathrm{Sp}(4,\mathbb{C}) < \mathrm{SL}(4,\mathbb{C})$ we get that any $g \in \mathrm{Sp}(4,\mathbb{C})$ preserves $B$. If $\mathcal{E}$ denotes the canonical basis of $\mathbb{C}^4$, since $g$ preserves the standard symplectic form on $\mathbb{C}^4$, it must fixes the element

$$\sigma := e_1 \wedge e_3 + e_2 \wedge e_4 \in \Lambda^2\,\mathbb{C}^4\,.$$

It easy to see that $B(\sigma,\sigma) \neq 0$. This means that the $\mathrm{Sp}(4,\mathbb{C})$-action preserves the decomposition

$$\Lambda^2\,\mathbb{C}^4 = \langle\sigma\rangle \oplus \langle\sigma\rangle^\perp$$

and also the restriction of $B$ to $\langle\sigma\rangle^\perp$. This implies that we get a smooth homomorphism $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$ which induces the desired isomorphism between the associated Lie algebras.

## Exercise 5

Let $M$ be a smooth manifold and let $X, Y$ be two complete vector fields on $M$. Prove that the following are equivalent

- The vector fields commute at every point of $M$, that is $[X, Y] = 0$.
- If we denote by $\Phi^X : \mathbb{R} \times M \to M$ (resp. $\Phi^Y$) the flow associated to the vector field $X$ (resp. $Y$) it holds $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$ for every $t, s \in \mathbb{R}$.

**Solution:** We are going to prove only one direction. Assume that the flows commute, that means

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$$

for every $t, s \in \mathbb{R}$. We are going to show that $[X, Y]_p = 0$ at every point $p \in M$.

By definition recall that it holds

$$\begin{cases} \Phi_t^X(p) & = p \\ \frac{d}{dt}|_{t=0}\Phi^X(t) & = X_{\Phi_t^X(p)}, \quad \text{for every } t \in \mathbb{R}. \end{cases} \tag{2}$$

This is a complicated way to say that $\Phi_t^X$ is the unique integral curve associated to $X$ and passing through $p$ a time $t = 0$. We are going to use Equation (2) to compute $Xf$ for any smooth function $f$ on $M$.

Let $f \in C^\infty(M)$. For every point $p \in M$ the value $(Xf)(p)$ can be computed as it follows

$$(Xf)(p) = \frac{d}{dt}|_{t=0}(f \circ \Phi_t^X(p)).$$

In the same way it holds

$$(Y(Xf))(p) = \frac{d}{ds}|_{s=0}((Xf) \circ \Phi_s^Y(p)) = \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0}(f \circ \Phi_t^X \circ \Phi_s^Y(p)). \tag{3}$$

If we now exchange the role of $X$ and $Y$ in the computation above we get

$$(X(Yf))(p) = \frac{d}{dt}|_{t=0}((Yf) \circ \Phi_t^X(p)) = \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}(f \circ \Phi_s^Y \circ \Phi_t^X(p)). \tag{4}$$

By Schwarz theorem and by the commutativity of the flows we get

$$0 = X(Yf)(p) - Y(Xf)(p) = ([X, Y](f))(p)$$

for every $p \in C^\infty(M)$. Since $f \in C^\infty(M)$ was arbitrary, it must hold $[X, Y]_p = 0$ for every $p \in M$ and the claim follows.