Exercise Sheet 6

Exercise 1

Let G be a Lie group and let $H < G$ be a closed subgroup. Show that G/H admits a structure of smooth manifold such that the natural action

$$\theta : G \times G/H \to G/H, \quad \theta(g, xH) = gxH$$

is smooth and the map

$$p : G \to G/H, \quad p(g) = gH$$

is a smooth fibration.

Exercise 2

Let $\mathcal{H}_{n+1}$ the space of homogeneous polynomials in the variables X, Y of degree equal to n with complex coefficients. Define a representation

$$\pi_{n+1} : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(\mathcal{H}_{n+1}), \quad (\pi_{n+1}(g)P)(X, Y) := P(g^{-1}(X, Y)),$$

where $g^{-1}(X, Y) := g^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}$.

Compute the set of weights of $\pi_{n+1}|_B$, where

$$B := \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} | a > 0, b \in \mathbb{R} \}.$$

Which are the associated weight spaces?

Exercise 3

Show that a connected Lie group is abelian if and only if its Lie algebra $\mathfrak{g}$ is abelian using Lemma II.59(i) applied to the adjoint representation.

Exercise 4

Let $\mathcal{L}$ be a finite dimensional $\mathbb{R}$-vector space and let $\Gamma < \mathcal{L}$ be a discrete subgroup. Show that there exist $e_1, \ldots, e_r$ linearly independent in $\mathcal{L}$ such that

$$\Gamma = \mathbb{Z}\, e_1 + \ldots + \mathbb{Z}\, e_r.$$