Exercise Sheet 6

Exercise 1

Let G be a Lie group and let $H < G$ be a closed subgroup. Show that G/H admits a structure of smooth manifold such that the natural action

$$\theta : G \times G/H \to G/H, \quad \theta(g, xH) = gxH$$

is smooth and the map

$$p : G \to G/H, \quad p(g) = gH$$

is a smooth fibration.

Solution: From the lecture we know that there exists a suitable neighborhood $U \subset \mathfrak{g}$ of the origin such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. Denote by $\mathfrak{h} = \mathrm{Lie}(H)$ the Lie algebra associated to H. Choose any complement $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$ as vector spaces. Define

$$V := U \cap \mathfrak{l}.$$

Since $V \cap \mathfrak{h} = \{0\}$ it is immediate to verify that $p \circ \exp|_V : V \to G/H$ is a homeomorphism onto the image. This gives us a local chart around the point $H \in G/H$. We can get an atlas by suitably translating this chart by the natural action of G on G/H. This gives us back an atlas such that each change of coordinate charts is smooth (since the multiplication in G is smooth). For more details about this part see Theorem 3.58 in *Foundations of differentiable manifolds and Lie groups* by Warner.

To verify that is a fibration notice that the fibers of the map $p : G \to G/H$ can be identified using the left translation by a suitable element of G. Notice that if H is not compact we cannot apply standard results in fibration theory since the projection p may be not proper.

Exercise 2

Let $\mathcal{H}_{n+1}$ be the space of homogeneous polynomials in the variables X, Y of degree equal to n with complex coefficients. Define a representation

$$\pi_{n+1} : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(\mathcal{H}_{n+1}), \quad (\pi_{n+1}(g)P)(X, Y) := P(g^{-1}(X, Y)),$$

where $g^{-1}(X, Y) := g^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}$.

Compute the set of weights of $\pi_{n+1}|_B$, where

$$B := \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} | a > 0, b \in \mathbb{R} \}.$$

Which are the associated weight spaces?

Solution: Recall that a weight for a representation is a homomorphism

$$\chi : G \to \mathbb{C}^{\times}$$

such that
$$V_\chi := \{v \in V | \pi(g)v = \chi(g)v, \forall g \in G\} \neq (0).$$
Hence we are looking for homomorphisms
$$\chi : B \to \mathbb{C}^\times$$
such that $V_\chi \neq (0)$. Recall that every homomorphism $\chi : B \to \mathbb{C}^\times$ induces a homomorphism $\overline{\chi} : B/[B,B] = B^{ab} \to \mathbb{C}^\times$. Notice now that
$$B^{ab} \cong \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a > 0\}$$
and hence any unipotent element will act trivially for a given weight. We need only to care about of the subgroup
$$\{\pi_{n+1} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a > 0\} = \{ \begin{pmatrix} a^{-n} & 0 & \dots & 0 & 0 \\ 0 & a^{-n+2} & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & a^{n-2} & 0 \\ 0 & 0 & \dots & 0 & a^n \end{pmatrix} | a > 0\}.$$

We get immediately that
$$\chi_i \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{2i-n}$$
is homomorphism, but only for $i = n$ the character χ_n is a weight for π_{n+1} whose space is given by $V_n := \langle Y^n \rangle$.

Exercise 3

Show that a connected Lie group is abelian if and only if its Lie algebra $\mathfrak{g}$ is abelian using Lemma II.59(i) applied to the adjoint representation.

Solution: Let
$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \quad \mathrm{Ad}(g) := D_e\mathrm{Int}(g)$$
be the adjoint representation. Assume that G is abelian and connected. By assumption we have that $gh = hg$ for every $g, h \in G$ or equivalently $ghg^{-1} = h$ that means that the homomorphism $\mathrm{Int}(g)$ acts trivially. In particular we have that $\mathrm{Int}(g) = \mathrm{id}_G$ for every $g \in G$. Hence we have that $\mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ for every $g \in G$.

Take $Y \in \mathfrak{g}$ and consider
$$S_Y := \{g \in G : \mathrm{Ad}(g)(Y) = Y\}.$$
By what we have said so far, it holds $S_Y = G$. In particular $\mathfrak{s}_Y = \mathrm{Lie}(S_Y) = \mathfrak{g}$, hence
$$\mathfrak{s}_Y = \{X \in \mathfrak{g} : \mathrm{ad}(X)(Y) = 0\} = \mathfrak{g},$$
from that we deduce that $\mathfrak{g}$ is abelian by the arbitrary choice of Y.

By reversing the argument and by the connectedness of G we get the statement.

Exercise 4

Let $\mathcal{L}$ be a finite dimensional $\mathbb{R}$-vector space and let $\Gamma < \mathcal{L}$ be a discrete subgroup. Show that there exist $e_1, \ldots, e_r$ linearly independent in $\mathcal{L}$ such that

$$\Gamma = \mathbb{Z}\, e_1 + \ldots + \mathbb{Z}\, e_r.$$

Solution: Let V_Γ be the $\mathbb{R}$ vector subspace generated by Γ, that is the smallest $\mathbb{R}$-vector subspace of $\mathcal{L}$ containing Γ. Denote by $r = \dim_{\mathbb{R}} V_\Gamma$. The exercise is equivalent to find a basis of V_Γ such that

$$\Gamma = \mathbb{Z}\, e_1 + \ldots + \mathbb{Z}\, e_r.$$

Fix a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of $\mathcal{L}$ so that we can identify $\mathcal{L}$ with $\mathbb{R}^n$ and consider the scalar product given by $\langle v_i, v_j \rangle = \delta_{ij}$. Denote by $|\cdot|$ the norm induced by the scalar product.

We are going to show the claim by induction on r. Assume $r = 1$. Hence V_Γ is a line and we need to show that there exist an element e_1 such that $V_\Gamma = \mathbb{Z}\, e_1$. Define e_1 as one of the two possible vectors such that

$$|e_1| := \min\{|v| : v \in \Gamma\}.$$

The existence of e_1 is guaranteed by the discreteness of Γ. It is easy to verify e_1 is the desired vector (you can check that it is a direct consequence of the minimality assumption).

Assume now that the statement is true for $r - 1$. We want to show it in the case of dimension equal to r. Define again e_1 as one of the possible vector such that

$$|e_1| := \min\{|v| : v \in \Gamma\}.$$

It is easy to see that $\mathbb{Z}\, e_1$ is a subgroup of Γ. Consider now the space $E_1 = \mathbb{R}\, e_1$ and the quotient $p : V_\Gamma \to V_\Gamma / E_1$. The image $p(\Gamma)$ of Γ in this quotient is still a lattice and hence by induction there exist $e_2, \ldots, e_r \in \Gamma$ such that $p(\Gamma) = \mathbb{Z}\, p(e_2) + \ldots + \mathbb{Z}\, p(e_r)$. To conclude you can verify that $e_1, \ldots, e_r$ satisfy the claim.