INTRODUCTION TO LIE GROUPS

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Abstract. These notes encompass basic material on topological groups, the Lie correspondence and some structure theory of Lie groups.

They are based on notes of A. Iozzi for her Lie groups course taught at the University of Pennsylvania, the University of Maryland and ETH Zurich over several years. Their origin lies in a course taught by R. Zimmer at the University of Chicago in 1985. In their present form, these notes were taken, modified and typeset by S. Tornier during the course "Lie Groups I" taught by M. Burger at ETH Zurich in 2013.

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Introduction

The origin of Lie groups lies in trying to solve certain equations. In Galois Theory, one learns that the solvability of polynomial equations is connected to the structure of the Galois group associated to the particular equation. Motivated by this, people worked towards a similar theory for differential equations, resulting in the theory of Lie groups. It did not fulfill the expectations but turned out to be extremely fruitful for other directions. For instance, Lie groups and their homogeneous spaces constitute a playground to test conjectures in algebraic topology on and provide a wealth of interesting manifolds. In number theory, a proof of the classical Fermat problem is currently unthinkable without a thorough development of the theory of Lie groups and automorphic forms. In geometry, Lie groups arise as groups of structure-preserving maps, thus providing insight into interesting, highly symmetric geometries. There will be a more detailed motivation of the subject once we have developed some foundations. A good historical account of the theory of Lie groups is [Bor01].

1. Topological Groups

1.1. **Topological Groups and Examples.** We start by fixing some notation. Given a group G, the neutral element is denoted $e \in G$. The product is written $G \times G \to G,\ (a, b) \mapsto a \cdot b$ and the inverse $G \to G,\ g \mapsto g^{-1}$.

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Definition 1.1. Let G be a group. A topology $\mathcal{T} \subseteq \mathcal{P}(G)$ endows G with the structure of a *topological group* if the product map $G \times G \to G$ and the inverse map $G \to G$ are continuous.

In the above definition, $G \times G$ is endowed with the product topology. Imagine playing the piano, the left hand taking care of topology, the right hand of algebra. Then topological group theory aims to play with both hands at the same time. Definition 1.1 has the following immediate consequences.

Remark 1.2. Retain the notation of Definition 1.1.

(i) The inverse map $i : G \to G, g \mapsto g^{-1}$ is continuous and bijective. Since its inverse $i^{-1} = i$ is also continuous, it is a homeomorphism.

(ii) Let $g \in G$. We define the left translation $L_g : G \to G$ by $x \mapsto gx$. This map is continuous as part of Definition 1.1 and bijective for every $g \in G$ by the group axioms. Its inverse is $L_{g^{-1}}$ since
$$(L_{g^{-1}} \circ L_g)(x) = L_{g^{-1}}(gx) = g^{-1}gx = (g^{-1}g)x = x$$
by associativity of the product. Again, these maps are actually homeomorphisms. Therefore, a topological group "looks locally everywhere the same". Whatever happens near the identity element $e \in G$ is replicated at $g \in G$ by the homeomorphism L_g.

Analogously, right multiplication $R_g : G \to G$ is defined by $x \mapsto xg$.

(iii) ("If you have a topological group, you (may) have many.") Let $H \leq G$ be a subgroup of G. Then H is a topological group when endowed with the induced topology.

(iv) Let G_1 and G_2 be topological groups and $h : G_1 \to G_2$ a homomorphism, i.e. $h(xy) = h(x)h(y)\ \forall x, y \in G_1$. Then h is continuous if and only if it is continuous at the identity $e \in G_1$ (or any other single point).

Example 1.3. We proceed by giving examples.

(i) Any group G with the discrete topology is a topological group. In this case, all subsets are open and thus all maps from G into any other topological space are continuous. Despite their seemingly dull definition, discrete groups constitute the most important examples in certain contexts.

(ii) The pair $(\mathbb{R}^n, +)$, equipped with the Euclidean topology, is a commutative topological group. It needs to be verified, that the maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (x, y) \mapsto x + y$ and $\mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto -x$ are continuous.

(iii) The multiplicative groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ of the fields $\mathbb{R}$ and $\mathbb{C}$ respectively, equipped with the topologies induced from the respective Euclidean topologies, are topological groups.

When it comes to the structure theory of Lie groups, Examples (ii) and (iii) will be considered elementary and left without further analysis.

(iv) If $\mathbb{F}$ is a field, $M_{n,m}(\mathbb{F})$ denotes the vector space of $n \times m$ matrices, i.e. matrices with n rows and m columns. Via the identification of $M_{n,m}(\mathbb{R})$ with $\mathbb{R}^{nm}$, the space $M_{n,m}(\mathbb{R})$ obtains a natural topology. It is well-known that $\mathrm{GL}(n, \mathbb{R}) := \{X \in M_{n,n}(\mathbb{R}) \mid \det(X) \neq 0\}$ is a group for the matrix product, with identity element $\mathrm{Id}_n = \mathrm{diag}(1, \dots, 1)$. We equip $\mathrm{GL}(n, \mathbb{R})$ with the topology induced from $M_{n,n}(\mathbb{R})$. This turns $\mathrm{GL}(n, \mathbb{R})$ into a topological group:

   (a) The matrix product $M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ (A, B) \mapsto A \cdot B$ is continuous: The i,j-entry of $A \cdot B$ is given by $(A \cdot B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$, hence is a polynomial function in the entries of A and B, thus continuous. The reader is encouraged to use this to write down a full proof of the continuity of the matrix product restricted to $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$.

(b) As to the inverse, let $X \in \mathrm{GL}(n,\mathbb{R})$. Then the i,j-entry of X^{-1} is given by $(X^{-1})_{ij} = \det C_{ji}/\det X$ where C_{ij} is the ij-cofactor of X obtained by deleting the i-th row and the j-th column of X and multiplying by $(-1)^{i+j}$. Since the determinant function is a polynomial in the respective matrix entries, it follows that inversion $\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$, $X \mapsto X^{-1}$ is continuous.
Also using the continuity of the determinant function, one shows that $\mathrm{GL}(n,\mathbb{R})$ is open in $M_{n,n}(\mathbb{R})$ and thus receives the structure of a manifold. In fact, $\mathrm{GL}(n,\mathbb{R})$ is a fundamental example of a Lie group.

(v) In Example (iv), the field $\mathbb{R}$ could have been replaced by any other topological field, such as $\mathbb{Q}_p$, the field of p-adic numbers, which constitutes an important such example. In fact, the $\mathrm{GL}(n,-)$-construction is very flexible.

For the next examples, we review some basic notions from general topology, see e.g. [Mun00]. To economically define topologies, one uses *bases* and *subbases.*

(GT1) Let X be a set. A basis of a topology $\mathcal{T} \subseteq \mathcal{P}(X)$ is a family $\mathcal{B} \subseteq \mathcal{T}$ such that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Consider for instance the case of $X = \mathbb{R}^n$ with the Euclidean topology $\mathcal{T}$. The whole list of open sets in $\mathbb{R}^n$ is uncountable. However, it can be proven that the set $\mathcal{B} := \{B_r(x) \mid r \in \mathbb{Q}_{\geq 0}, x \in \mathbb{Q}^n\}$ is a countable basis for $\mathcal{T}$, using the following criterion.

(GT2) A family $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for a unique topology on X if and only if the following two conditions are satisfied: (i) $X = \bigcup_{Y \in \mathcal{B}} Y$, (ii) If $B_1, B_2 \in \mathcal{B}$ are such that $B_1 \cap B_2 \neq \emptyset$ then for every $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. The topology on X with basis $\mathcal{B}$ then consists of all possible unions of elements of $\mathcal{B}$.

An even more economical way to define a topology is to give a *subbasis.*

(GT3) A subbasis of a topology $\mathcal{T}$ is a family of sets $\mathcal{S} \subseteq \mathcal{T}$ such that the family of sets $\mathcal{B} \subseteq \mathcal{T}$ obtained by taking all finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$.

(GT4) A family of sets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a subbasis for a unique topology on X if and only if $\mathcal{S}$ is not empty. The topology on X with subbasis $\mathcal{S}$ is given by (GT3) and (GT2).

Finally, we recall what locally compact and Hausdorff mean.

(GT5) A topological space X is Hausdorff if any two distinct points have disjoint neighbourhoods. It is locally compact if for all $x \in X$ and for every neighbourhood V of x there is a compact neighbourhood W of x such that $x \in W \subseteq V$.

Later on, we will check whether given examples satisfy the conditions of (GT5) but for now we proceed by giving more examples.

Recall that if X and Y are topological spaces, the compact-open topology on the set $C(X,Y) := \{f : X \to Y \mid f \text{ is continuous}\}$ is defined by the subbasis:

$$\mathcal{S} = \{S(C,U) \mid C \subseteq X \text{ compact, } U \subseteq Y \text{ open}\}.$$

where $S(C,U) := \{f : X \to Y \mid f(C) \subseteq U\}$. This is one of many topologies $C(X,Y)$ may be endowed with and the one for which the Arzelà -Ascoli theorem is formulated.

(vi) Let X be a compact Hausdorff topological space and consider

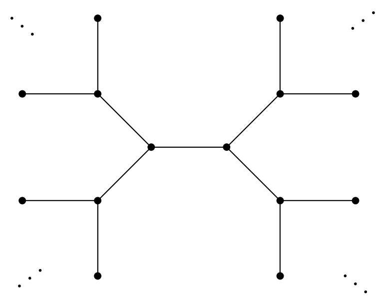
$$\mathrm{Homeo}(X) := \{f : X \to X \mid f \text{ is a homeomorphism}\}.$$

This set is a group under composition of maps. Inverses are the natural inverses of bijective maps. We can endow $\mathrm{Homeo}(X) \subseteq C(X,X)$ with the topology induced by the compact-open topology on $C(X,X)$ and thus turn it into a topological group. The proof of this claim does use the assumptions on X which may seem unnatural at first. If X is a locally compact, non-compact Hausdorff topological space, $\mathrm{Homeo}(X)$ need not be a topological group with the compact-open topology. However it is, if X is locally connected in addition. This includes all manifolds. The same holds true, if X is a proper metric space, i.e. a metric space in which closed balls of finite radius are compact.

(vii) Let $G_\alpha,\ \alpha \in A$ be a family of topological groups. Then the set $\prod_{\alpha\in A} G_\alpha$ is a topological group with pointwise composition and the product topology. This seemingly easy construction is not to be underestimated. As an example, the topological group $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\,\mathbb{Z}$ is compact by Tychonoff's theorem where $\mathbb{Z}/2\,\mathbb{Z}$ is endowed with the discrete topology.

For the next example, recall that if (X,d) is a metric space, an *isometry* of X is a bijection $f: X \to X$ which preserves distances, i.e. $d(f(x),f(y)) = d(x,y)\ \forall x,y \in X$. Note, that there may be various definitions of an isometry in various contexts. For instance, a map $f: X \to X$ which preserves distances is injective but not necessarily surjective. We therefore require surjectivity, to turn the set $\mathrm{Iso}(X)$ of all isometries of X into a group.

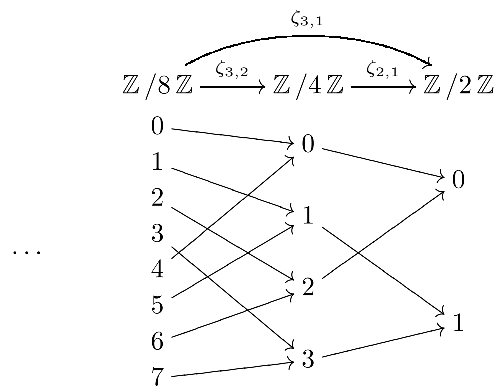
(viii) Let (X,d) be a proper metric space. We may endow $\mathrm{Iso}(X) \subseteq \mathrm{Homeo}(X)$ with the induced topology, i.e. the compact-open topology. For instance, if $X = (\mathbb{R}^n, d)$ and d is the Euclidean distance, then $\mathrm{Iso}(X) = \mathrm{O}(n,\mathbb{R}) \ltimes \mathbb{R}^n$. We may also choose X to be a regular tree T_d of finite valency d, e.g. T_3:



The isomety group of T_3 is gigantic and has no Lie group structure in whatever sense, thus lies outside the realm of Lie theory.

(ix) We may consider $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$ with the topology induced by the compact-open topology on $\mathrm{Homeo}(\mathbb{R}^n)$. The latter coincides with the topology induced from the Euclidean topology on $\mathbb{R}^{n\cdot n}$ via $\mathrm{GL}(n,\mathbb{R}) \subseteq M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n\cdot n}$. Therefore, if $A, B \in \mathrm{GL}(n,\mathbb{R})$ are entry-wise close in the Euclidean sense, they are also uniformly close as maps on compact subsets of $\mathbb{R}^n$ and conversely. Depending on the context, one or the other description may be more useful. In fact, it is not completely useless to think of a linear bijection as a homeomorphism.

(x) In this example, we discuss the notion of inverse limit by looking at its simplest and most important example. Let p be a prime number. Recall that $(\mathbb{Z}/p^n\,\mathbb{Z}, +)$ $(n \in \mathbb{N})$ denotes the additive group of integers modulo p^n. Endowed with the discrete topology, $\mathbb{Z}/p^n\,\mathbb{Z}$ is a compact topological group. For $n \geq m$ $(n,m \in \mathbb{N})$, there is a natural, surjective group homomorphism $\zeta_{n,m} : \mathbb{Z}/p^n\,\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\,\mathbb{Z}$, namely reduction modulo p^m. For $p = 2$, these homomorphisms may be visualized as follows:



Now the projective, or inverse limit of the system $(\mathbb{Z}/p^n\mathbb{Z}, \zeta_{n,m})_{n,m}$, written $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the set of all sequences $(x_n)_{n\in\mathbb{N}}$, where $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ compatible with the maps $\zeta_{n,m}$ $(n \geq m \geq 1)$, formally:

$$\varprojlim \mathbb{Z}/p^n\mathbb{Z} = \left\{ (x_n)_n \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} \,\middle|\, \forall n \geq m \geq 1 : \ \zeta_{n,m}(x_n) = x_m \right\}.$$

In the above picture, these sequences arise by backtracking an element of $\mathbb{Z}/2\mathbb{Z}$ along the tree-like structure produced by the homomorphisms $\zeta_{n,m}$. The set $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is actually a closed, thus compact subgroup of $\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$, called the p-adic integers and usually denoted by $\mathbb{Z}_p$. They were introduced by Hensel in the following context: It is well-known that the polynomial $x^2+1 \in \mathbb{Z}[x]$ has no roots over the real numbers. However, it is satisfiable in $\mathbb{Z}/p^n\mathbb{Z}$ for each n and there are compatible such solutions, i.e. it has a root in $\mathbb{Z}_p$ and this space may be considered as a space of possible solutions just as the reals may. This is well explained at [BS86, I.1]. To shed some more light on this interesting compact topological group, note that there is a natural injective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_p$ which has dense image. Hence $\mathbb{Z}_p$ is, in a sense, a completion/compactification of $\mathbb{Z}$.

This is actually an embryo example which has quite some offspring. For instance, $\mathbb{Z}_p$ may naturally be turned into a ring and the assumption that p be prime is only needed to avoid zero-divisors in this ring. Also, the topological groups $\mathbb{Z}_p$ are all totally disconnected and thus cannot occur as closed subgroups of $\mathrm{GL}(n,\mathbb{R})$. In fact, they all look like the Cantor set.

(xi) Resuming our discussion of the fundamental example $\mathrm{GL}(n,\mathbb{R})$, there are the following three important subgroups.

(a) The subgroup

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \,\middle|\, \forall\, 1 \leq i \leq n : \ \lambda_i \in \mathbb{R} - \{0\} \right\}$$

of diagonal matrices does not get much emphasis in linear algebra because diagonal matrices are very basic linear maps but will be totally essential in Lie theory. It is isomorphic to $\mathbb{R}^{*n}$ and a closed, abelian subgroup of $\mathrm{GL}(n,\mathbb{R})$.

(b) The subgroup

$$N = \left\{ (x_{ij})_{i,j} = \begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \forall\, 1 \le i \le n: \ x_{ii} = 1, \\ \forall\, 1 \le j < i \le n: \ x_{ij} = 0 \end{array} \right\}$$

of unipotent upper-triangular matrices is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ and not abelian, unless $n \le 2$.

(c) The orthogonal group

$$K = \mathrm{O}(n, \mathbb{R}) = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T X = \mathrm{Id}_n\}$$

is the group of linear isometries of $\mathbb{R}^n$ and a compact, hence closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. It is not abelian, unless $n \le 2$.

These subgroups are very important because they display very different behaviours of subgroups of $\mathrm{GL}(n, \mathbb{R})$. Note also the following: The Gram-Schmidt orthonormalization algorithm can be reformulated in the following way: Every $g \in \mathrm{GL}(n, \mathbb{R})$ is in a unique way a product $g = kan$ where $k \in K$, $a \in A$ and $n \in N$. In general Lie theory, this is called *Iwasawa decomposition*. In particular, it says that $\mathrm{GL}(n, \mathbb{R})$ is homeomorphic to the topological space $K \times A \times N$ to the effect that its algebraic topology (think fundamental group, homology) is concentrated in K.

(xii) Consider the following bilinear form on $\mathbb{R}^n$:

$$B(x, y) = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n} x_j y_j = x^T \begin{pmatrix} -\mathrm{Id}_p & \\ & \mathrm{Id}_q \end{pmatrix} y$$

where $n = p + q$, i.e. the second sum above has q terms. This is the symmetric, non-degenerate bilinear form of signature (p, q) with respect to the standard basis (recall Silvester's theorem). The group

$$\mathrm{O}(p, q) = \left\{ X \in \mathrm{GL}(n, \mathbb{R}) \;\middle|\; X^T \begin{pmatrix} -\mathrm{Id}_p & \\ & \mathrm{Id}_q \end{pmatrix} X = \begin{pmatrix} -\mathrm{Id}_p & \\ & \mathrm{Id}_q \end{pmatrix} \right\}$$

of invertible linear transformations preserving B is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ and another important example of a Lie group.

Having discussed a lot examples of topological groups, we now turn to the question whether they are locally compact or even compact. These are important properties. The following result from general topology will be useful:

(GT6) Let X be a locally compact Hausdorff topological space. Then every closed and every open subset of X is locally compact with respect to the induced topology.

(i) Let G be a discrete group. Then G is locally compact; and it is compact if and only if G is finite. In fact, an arbitrary Hausdorff topological group is finite if and only if it is discrete and compact; an easy yet powerful criterion.

(ii) Let $G = (\mathbb{R}^n, +)$. Then G is locally compact but not compact unless $n = 0$.

(iii) The groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ are locally compact as open subspaces of $\mathbb{R}$ and $\mathbb{C}$ respectively by (GT6) but not compact.

(iv) The group $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n \cdot n}$ and thus locally compact by (GT6). Hence also all the closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ given in example xi are locally compact. However, $\mathrm{GL}(n, \mathbb{R})$ is not compact for several reasons:

(a) If $\mathrm{GL}(n, \mathbb{R}) \subsetneq \mathbb{R}^{n \cdot n}$ was compact, it would be closed. Since $\mathrm{GL}(n, \mathbb{R})$ is also open, this would contradict the connectedness of $\mathbb{R}^{n \cdot n}$.

(b) If $\mathrm{GL}(n,\mathbb{R})$ was compact, then so would $\det(\mathrm{GL}(n,\mathbb{R})) = \mathbb{R} - \{0\}$ be.

(c) By Heine-Borel, a subset of $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n\cdot n}$ is compact if and only if it is closed and bounded; and $\mathrm{GL}(n,\mathbb{R})$ is closed but unbounded.

(d) The subset $\{\mathrm{Id}_n + tE_{1,n} \mid t \in \mathbb{R}\} \subset \mathrm{GL}(n,\mathbb{R})$, where $E_{1,n} = (\delta_{i,1}\delta_{j,n})_{ij}$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ which with the induced topology is isomorphic to $(\mathbb{R}, +, \mathcal{T}_{\mathrm{eucl}})$ as a topological group. Since the latter is not compact, $\mathrm{GL}(n,\mathbb{R})$ cannot be compact either.

(vi) For any manifold X of positive dimension, $\mathrm{Homeo}(X)$ is not locally compact, e.g. $\mathrm{Homeo}(S^1)$.

(vii) Let G_α $(\alpha \in A)$ be a family of topological groups. Then $G = \prod_{\alpha\in A} G_\alpha$ is compact if and only if all G_α are compact; and locally compact if and only if all G_α are locally compact and all except possibly finitely many G_α are actually compact.

(viii) Let (X, d) be a metric space. The Arzelà -Ascoli theorem says that a subset of $\mathcal{F} \subseteq \mathrm{Iso}(X)$ has compact closure if and only if it is equicontinuous and for every $x \in X$, the set $\{f(x) \mid f \in \mathcal{F}\} \subseteq X$ has compact closure. Since any family of isometries is equicontinuous it suffices to check the second condition. This immediately implies that $\mathrm{Iso}(X)$ is compact if X is compact. It can be shown that $\mathrm{Iso}(X)$ is always locally compact. The reader may provide such a proof for proper metric spaces. As an example, recall that $\mathrm{Iso}(S^n) = \mathrm{O}(n,\mathbb{R})$ is compact.

(xii) The topological group $\mathrm{O}(p, q)$ is locally compact as a closed subspace of $\mathrm{GL}(n,\mathbb{R})$ by (GT6). It is compact if and only $p = 0$ or $q = 0$. To get an idea of why this is true, consider $\mathrm{O}(1, 1) \leq \mathrm{GL}(2,\mathbb{R})$. The set of matrices

$$\left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \middle|\, t \in \mathbb{R} \right\}$$

is contained in $\mathrm{O}(1, 1)$, basically because $\cosh^2 t - \sinh^2 t \equiv 1$. It is also a subgroup and as such isomorphic to $(\mathbb{R}, +)$ as a topological group; this is due to some addition theorems for cosh and sinh. Containing a copy of $\mathbb{R}$, the group $\mathrm{O}(1, 1)$ cannot be compact.

1.2. **Some (Miraculous) Facts about Topological Groups.** We recall some facts about the notion of connectedness of topological spaces. A topological space X is *connected* if it cannot be written as a disjoint union of two proper open sets. A subset of a topological space is connected if it is connected as a topological space with the induced topology. Recall that the closure of a connected subset is connected and that a continuous image of a connected space is connected. Finally, given a topological space X, the relation $x \sim y$ $(x, y \in X)$ if and only if $\{x, y\}$ is contained in a connected subset of X is an equivalence relation on X. Its equivalence classes are called connected components. They are maximal connected subsets; that is, if $C(x)$ denotes the connected component of $x \in X$ and A is a connected set containing $x \in X$, then $A \subseteq C(x)$.

Proposition 1.4. Let G be a topological group. Then the following statements hold.

(i) If $H \leq G$ is a subgroup then so is $\overline{H}$.

(ii) If $H \leq G$ is an open subgroup, then it is closed.

(iii) The connected component G_0 containing the identity $e \in G$ is a closed normal subgroup of G.

(iv) If G is connected and $V \in \mathcal{U}(e \in G)$ then $G = \bigcup_{n\geq 1}(V \cup V^{-1})^n$.

(v) If G is connected and $N \trianglelefteq G$ is a discrete normal subgroup, then N is contained in the center $\mathrm{Z}(G)$ of G.

Proof. Let G be a topological group. For part (i), recall that given topological spaces X and Y, a map $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets $A \subseteq X$ of X. Applying this to the multiplication map m and the inversion i, we obtain $m(\overline{H} \times \overline{H}) = m(\overline{H \times H}) \subseteq \overline{m(H \times H)} = \overline{H}$ and $i(\overline{H}) \subseteq \overline{i(H)} = \overline{H}$. Hence $\overline{H}$ is a subgroup of G as well.

For part (ii), let R be a complete set of representatives for the set $G/H = \{gH \mid g \in H\}$ of left cosets of H in G. Suppose $e \in R$. Then

$$G = \bigsqcup_{x \in R} xH = H \sqcup \bigsqcup_{x \in R - \{e\}} xH.$$

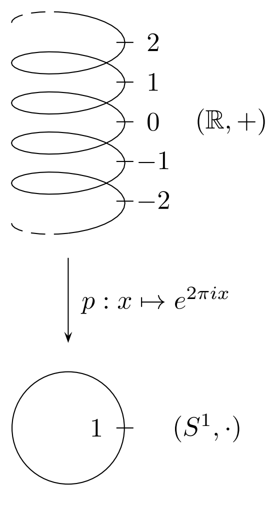
Therefore, the complement of H in G is open as a union of open sets, therefore H is also closed. As to part (iii), note first of all, that connected components are always closed, therefore so is G_0. Furthermore, the sets $m(G_0 \times G_0)$, $i(G_0)$ and $c_g(G_0) := \{gxg^{-1} \mid x \in G_0\}$ $(g \in G)$ are connected subsets of G containing the identity, hence they are contained in G_0 which is hence a normal subgroup of G.

As to part (iv), note that $H := \bigcup_{n \geq 1}(V \cup V^{-1})^n$ is a subgroup. Now, let U be an open neighbourhood of $e \in G$ contained in V. Then $H_U := \bigcup_{n \geq 1}(U \cup U^{-1})^n$ is a subgroup as well and as such is open, hence also closed by part (iii). Since G is connected, this implies $G = H_U \subseteq H$.

Finally, we prove part (v): If N is a discrete normal subgroup of G then for every $n \in N$, the continuous map $c_n : G \to G,\ g \mapsto gng^{-1}$ corestricts to a continuous map to N, carrying the discrete (totally disconnected) topology. Since, $c_n(G)$ is a connected set containing n, this implies $c_n(G) = n$, i.e. $n \in \mathrm{Z}(G)$. ☐

A sample application of Proposition 1.4 (v) is the following.

Remark 1.5. (Cultural). If G is a topological group which satisfies the conditions of covering theory, namely being connected, locally path-connected and semi-locally simply connected, then its universal cover $(\widetilde{G}, \tilde{e}, p : \widetilde{G} \to G)$ can be shown to be a topological group such that $p : \widetilde{G} \to G$ is a continuous homomorphism whose kernel $\ker p$ may be identified with $\pi_1(G)$. An example of this situation is the universal cover $(\mathbb{R}, 0, \exp(2\pi i -))$ of $(S^1, 1)$ where S^1 is considered as a subset of $\mathbb{C}$. Note that $\widetilde{G}$ being a cover of G implies that $\ker p$ is a discrete subgroup of $\widetilde{G}$. Further being normal as the kernel of a homomorphism, Proposition 1.4(v) implies that it is contained in the center of $\widetilde{G}$; in particular it is abelian. This is far from true for arbitrary topological spaces; for instance, the figure eight space $S^1 \vee S^1$ has a free group on two generators as fundamental group.



In addition to Proposition 1.4(iii), we make the following two remarks about the identity component of a topological group.

Remark 1.6. If G is a topological group, then G_0 is a (closed) normal subgroup of G by Proposition 1.4(iii). Hence the set $\pi_0(G)$ of connected components of G, which may be identified with G/G_0 has a natural group structure, namely $\pi_0(G) \cong G/G_0$. This group need not be abelian in general and is thus, in a sense, more complicated then $\pi_1(G)$ which is always abelian by Remark 1.5. If G is the homeomorphism group of a topological space, $\pi_0(G) \cong G/G_0$ is often called the *mapping class group* of that space.

We now check whether the examples of topological groups G we have discussed earlier are connected and try to determine G_0 as well as G/G_0. Obviously, for a connected group G, we have $G_0 = G$ and $G/G_0 = G$.

(i) If G is discrete, then G is totally disconnected, i.e. every point coincides with its connected component. In particular, $G_0 = \{e\}$ and $G/G_0 \cong G$.

(ii) The group $G = (\mathbb{R}^n, +)$ is connected, even path-connected. Thus $G_0 = \mathbb{R}^n$ and $G/G_0 \cong \{e\}$.

(iii) If $G = (\mathbb{R}^*, \cdot)$, then $G_0 = \mathbb{R}_{>0}$ and $G/G_0 \cong \mathbb{Z}/2\,\mathbb{Z}$. On the other hand, $G = (\mathbb{C}^*, \cdot)$ is connected, even path-connected.

(iv) The group $G = \mathrm{GL}(n, \mathbb{R})$ is not connected since its image $\mathbb{R} - \{0\}$ under the continuous determinant function is not. Moreover, one can show that $G_0 = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid \det X > 0\}$ and hence $G/G_0 \cong \mathbb{Z}/2\,\mathbb{Z}$.

(vi) The identity component of the homeomorphism group $\mathrm{Homeo}(X)$ of a compact Hausdorff space is tricky to determine even if X is a compact manifold. For instance, if $X = S^1$ then $\mathrm{Homeo(X)}_0$ can be shown to consist of the orientation-preserving homeomorphisms. The torus case $X = S^1 \times S^1$ is already considerably more complex as $\mathrm{Homeo}(X)/\mathrm{Homeo}(X)_0 \cong \mathrm{GL}(2, \mathbb{Z})$ (which is not abelian). This expression stems from the action of $\mathrm{Homeo}(X)$ on $\mathrm{H}_1(X) \cong \mathbb{Z}^2$. And higher genus surfaces are even more fascinating in this regard!

(vii) A product of topological groups is connected if and only if all its factors are connected.

(viii) There is not much to say about isomorphism groups of metric spaces in general.

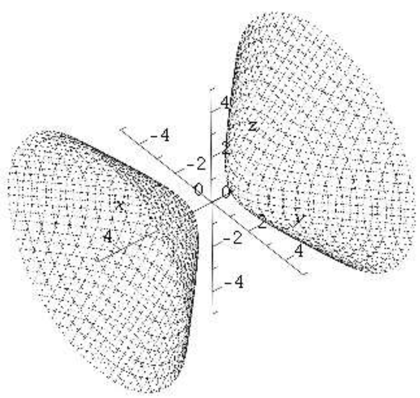
(x) The p-adic integers constitute a totally disconnected space as a subspace of a product of discrete, hence totally disconnected spaces.

(xi) The group $A \leq \mathrm{GL}(n, \mathbb{R})$ is isomorphic to $(\mathbb{R}^*)^n$, hence $A_0 \cong (\mathbb{R}_{>0})^n$. The subgroup N is connected. Finally, $K = \mathrm{O}(n, \mathbb{R})$ has identity component $\mathrm{O}(n, \mathbb{R})_0 = \mathrm{SO}(n, \mathbb{R})$; the reader is encouraged to find a good reason for why this is.

(xii) If $p, q \geq 1$, then $\mathrm{O}(p, q)$ has four connected components and one can show that $\mathrm{O}(p, q)/\mathrm{O}(p, q)_0 \cong \mathbb{Z}/2\,\mathbb{Z} \times \mathbb{Z}/2\,\mathbb{Z}$. To illustrate this, consider the case of $G = \mathrm{O}(1, 2)$ which preserves the set of points

$$X = \{x \in \mathbb{R}^3 \mid B_{1,2}(x, x) + 1 = 0\} \subset \mathbb{R}^3 .$$

These points form a two-sheeted hyperboloid:



The four connected components of G then correspond to the sign of the determinant and whether or not the two sheets of the above hyperboloid are interchanged.

1.3. **Haar Measure.** In this section we zoom in on a certain class of topological groups, namely locally compact Hausdorff ones, for which there is a rich structure to analyse, due to the existence of Haar measures. Alongside we introduce homogeneous spaces which constitute important examples of spaces on which topological groups, and later on Lie groups, act.

Let G be a group and let X be a locally compact Hausdorff topological space. We assume that G acts on X by homeomorphisms, i.e. there is an action $G \times X \to X$, $(g, x) \mapsto g_* x$ of G on X such that for all $g \in G$, the map $X \to X,\ x \mapsto g_* x$ is a homeomorphism. Further, let $C_{00}(X)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on X with compact support. This space is sometimes also denoted $C_c(X)$. Now, every $g \in G$ gives rise to an invertible linear transformation $\lambda(g) \in \mathrm{GL}(C_{00}(X))$ of $C_{00}(X)$ via

$$(\lambda(g)f)(x) = f(g_*^{-1}x) \quad (f \in C_{00}(X),\ x \in X).$$

The formula above contains g_*^{-1} instead of g_* so that the map $\lambda : G \to \mathrm{GL}(C_{00}(X))$ is a homomorphism. The idea of turning a complicated object, e.g. a homeomorphism as above, into an invertible linear map to which well-understood linear algebra applies has turned out to be extremely fruitful and in fact has lead to the branch of mathematics known as representation theory. In this context, λ is called the *left-regular representation* of G on $C_{00}(X)$.

Given the representation λ, we can consider its *contragredient* or *dual* representation: Every $g \in G$ gives rise to an invertible linear transformation $\lambda^*(g) \in \mathrm{GL}(\mathrm{L}(C_{00}(X), \mathbb{C}))$ of the algebraic dual $\mathrm{L}(C_{00}(X), \mathbb{C})$ of $C_{00}(X)$, consisting of $\mathbb{C}$-linear maps from $C_{00}(X)$ to $\mathbb{C}$, via

$$(\lambda^*(g)m)(f) = m(\lambda(g)^{-1}f) \quad (m \in \mathrm{L}(C_{00}(X), \mathbb{C}),\ f \in C_{00}(X)).$$

Again, the map $\lambda^* : G \to \mathrm{GL}(\mathrm{L}(C_{00}(X), \mathbb{C}))$ is a homomorphism.

By the Riesz Representation Theorem, certain elements of $\mathrm{L}(C_{00}(X), \mathbb{C})$ give rise to certain measures on X. Recall that $\Lambda \in \mathrm{L}(C_{00}(X), \mathbb{C})$ is *positive* if $\Lambda(f) \in \mathbb{R}$ whenever $f \in C_{00}(X)$ is real-valued and $\Lambda(f) \geq 0$ whenever $f \geq 0$.

Theorem (Riesz). Let $\Lambda \in \mathrm{L}(C_{00}(X), \mathbb{C})$ be a positive linear functional. Then there is a regular Borel measure μ on X which represents Λ in the sense that for all $f \in C_{00}(X)$:

$$\Lambda(f) = \int_X f(x)\ d\mu(x),$$

where the right hand side refers to the Lebesgue integral. An efficient introduction to this can be found at [Rud87, Ch. 1, 2].

One way to look at Riesz' Theorem is to say that every positive linear functional on $C_{00}(X)$ extends to a larger class of functions, e.g. $L^1(X)$, via the right hand side integral. As a sample application, note that the Lebesgue measure on $\mathbb{R}$ may be constructed by taking as Λ the Riemann integral.

Retain the notation of Riesz' Theorem. Then to $\lambda^*(g)\Lambda$ $(g \in G)$ corresponds a unique measure $g_*\mu$ whose relation with μ is given by $(g_*\mu(E)) = \mu(g^{-1}E)$ for all Borel sets $E \in \mathcal{B}(X)$ and $g \in G$. The reader is encouraged to verify this.

We now turn to the case where $X = G$; that is, G is a locally compact Hausdorff group, acting on itself by left translation. Remember that we have seen many examples of groups of this type.

Definition 1.7. A *left Haar measure* on G is a non-zero positive linear functional $m : C_{00}(G) \to \mathbb{C}$ which is invariant under left-translation, i.e. $\lambda^*(g)m = m$ for all $g \in G$. Equivalently, $m(\lambda(g)f) = m(f)$ for all $f \in C_{00}(X)$ and $g \in G$.

The same definition is useful with "left" replaced by "right". If μ is the regular Borel measure on G corresponding to m by Riesz' Theorem, then $\mu(g^{-1}E) = \mu(E)$ for all Borel sets $E \in \mathcal{B}(G)$.

At this point, we give three examples of Haar measures. More are to follow later.

Example 1.8.

(i) Let $G = (\mathbb{R}^n, +)$. Then the Lebesgue measure is a left Haar measure on G.

(ii) Let $G = (\mathbb{R}_{>0}, \cdot)$. In this case, the Lebesgue measure λ is not left-invariant. However, the map

$$m : C_{00}(X) \to \mathbb{C},\ f \mapsto \int_G f(x) \frac{d\lambda(x)}{x}$$

can be checked to be left-invariant and thus defines a Haar measure on G. Note that, since f has compact support, the function $f(x)/x$ is still integrable.

(iii) Let G be discrete. Then the rule $\mu(E) = \operatorname{card}(E)\ \forall E \subseteq G$ defines a left Haar measure on G.

It is a very important fact that a left Haar measure exists for any locally compact Hausdorff group. This was first proven by Haar for second-countable such groups in 1933 and was used to make significant progress on Hilbert's fifth problem.

Theorem 1.9 (Haar '33). On every locally compact Hausdorff group, there exists a left Haar measure which is unique up to positive scalar multiples.

The existence part of the proof is not exactly enlightening. Also, it has no further mathematical offspring whatsoever and we shall not give it here. A good version can be found at [Wei65]. We are, however, going to give a proof of the uniqueness part which is equally important and which gives us the opportunity to play around with the notion of a Haar measure.

We start with the following lemma for which we introduce the right-regular representation ϱ of G on $C_{00}(X)$ in analogy to the left-regular representation λ above:

$$(\varrho(g)f)(x) = f(xg) \quad (f \in C_{00}(X),\ x \in X).$$

Lemma 1.10. Let G be a locally compact Hausdorff group with left Haar measure m. Then the map $l : C_{00}(X) \to \mathbb{C},\ f \mapsto m(\overline{f})$ where $\overline{f}(x) := f(x^{-1})\ \forall x \in G$ defines a right Haar measure on G.

Proof. We have to show that $l(\varrho(g)f) = l(f)$ for all $g \in G$. Compute

$$\begin{aligned} l(\varrho(g)f) = m(\overline{\varrho(g)f}) &= \int_G f(x^{-1}g)\ d\mu(x) = \int_G f((g^{-1}x)^{-1})\ d\mu(x) \\ &= \int_G \overline{f}(g^{-1}x)\ d\mu(x) = m(\lambda(g)\overline{f}) = m(\overline{f}) = l(f). \end{aligned}$$

□

For the next lemma, recall that if $(X, \mathcal{B}(X), \mu)$ is a Borel measure space, the support of μ makes precise where the measure "lives" and is defined to be the set

$$\operatorname{supp}(\mu) := \{x \in X \mid \forall U \in \mathcal{U}(x) \text{ open}:\ \mu(U) > 0\}.$$

The openness assumption on the neighbourhoods is to make them measurable. There are in general non-measurable neighbourhoods.

Lemma 1.11. Let G be a locally compact Hausdorff group with left Haar measure m and associated regular Borel measure μ. Then

(i) $\operatorname{supp}\mu = G$, and
(ii) If $h \in C(G)$ satisfies $\int_G h(x)\varphi(x)\ d\mu(x) = 0\ \forall\varphi \in C_{00}(G)$ then $h \equiv 0$.

Proof. For part (i), note that since m is not the zero-measure, there is $f \in C_{00}(X)$ such that $m(f) > 0$. Then $\operatorname{supp} f$ is a compact set with $\mu(\operatorname{supp} f) > 0$. If there was $x \in G - \operatorname{supp}\mu$, there would be an open neighbourhood U of x with zero measure. But finitely many translates of U would cover K, hence $\mu(K) = 0$.

As to part (ii), we show that $h(e) = 0$. It will be clear how to adjust the argument to any other point. Let $\varepsilon > 0$. By continuity of h, there is an open neighbourhood V of $e \in G$ such that $|h(t) - h(e)| < \varepsilon$ for all $t \in V$. By Urysohn's Lemma, there is $\varphi \in C_{00}(G)$ such that $\varphi \geq 0,\ \varphi(e) > 0$ and $\operatorname{supp}\varphi \subset V$. Then

$$\begin{aligned}\left|\int_G h(t)\varphi(t)\ d\mu(t) - \int_G h(e)\varphi(t)\ d\mu(t)\right| &= \left|\int_V (h(t) - h(e))\varphi(t)\ d\mu(t)\right| \\ &\leq \varepsilon \int_V |\varphi(t)|\ d\mu(t)\end{aligned}$$

Applying the assumption, we obtain

$$|h(e)| \left|\int_V \varphi(t)\ d\mu(t)\right| \leq \varepsilon \int_V |\varphi(t)|\ d\mu(t).$$

In view of part (i) and the properties of φ, this implies $|h(e)| \leq \varepsilon$ for all $\varepsilon > 0$, hence $h(e) = 0$. $\square$

We are now in a position to prove the uniqueness part of Theorem 1.9.

Proof. (Theorem 1.9, uniqueness). Let m and m' be two left Haar measures an G. Define $n : C_{00}(X) \to \mathbb{C}$ by $f \mapsto m'(\overline{f})$ where $\overline{f}(x) := f(x^{-1})\ \forall x \in G$. Then n is a right Haar measure by Lemma 1.10. If ν is the regular Borel measure corresponding to n, we have $m(f)n(g) = m(f)\int_G g(y)\ d\nu(y)$ for all $f, g \in C_{00}(G)$ and hence by right-invariance of ν and Fubini's theorem:

$$\begin{aligned}m(f)n(g) &= \int_G f(t)\int_G g(yt)\ d\nu(y)\ d\mu(t) \\ &= \int_G\int_G f(t)g(yt)\ d\mu(t)\ d\nu(y) \\ &= \int_G\int_G f(y^{-1}t)g(t)\ d\mu(t)\ d\nu(y) \\ &= \int_G\int_G f(y^{-1}t)\ d\nu(y)\ g(t)\ d\mu(t).\end{aligned}$$

If $f \in C_{00}(G)$ satisfies $m(f) \neq 0$ we define

$$m_f : G \to \mathbb{C},\ t \mapsto \frac{1}{m(f)}\int_G f(y^{-1}t)\ d\nu(y).$$

The above equality then reads

$$n(g) = \int_G m_f(t)g(t)\ d\mu(t).$$

That is, whenever $f_1, f_2 \in C_{00}(G)$ are such that $m(f_i) \neq 0$ ($i \in \{1, 2\}$) we obtain

$$\int_G (m_{f_1}(t) - m_{f_2}(t))g(t)\ d\mu(t) = 0.$$

Since $m_{f_1} - m_{f_2} : G \to \mathbb{C}$ is continuous, Lemma 1.11 implies $m_{f_1} \equiv m_{f_2}$. Therefore $c := m_f(e)$ does not depend on the choice of such an f and hence

$$m(f) \cdot c = m(f)m_f(e) = \int_G f(y^{-1})\ d\nu(y) = n(\overline{f}) = m'(f)$$

for all $f \in C_{00}(G)$ satisfying $m(f) \neq 0$. If $m(f) = 0$, one can use this to conclude that $m'(f) = 0$ as well. Hence the assertion. $\square$

We are now going to exploit the above uniqueness result: Note that if G is a locally compact Hausdorff group with left Haar measure m and $\alpha \in \mathrm{Aut}(G)$ is an automorphism of G, i.e. a bijective homomorphism which is also a homeomorphism, then $m' : C_{00}(G) \to \mathbb{C},\ f \mapsto m(f \circ \alpha)$ is a left Haar measure on G as well. By uniqueness, there is a positive real number $\mathrm{mod}_G(\alpha)$ such that $m' = \mathrm{mod}_G(\alpha)m$, in formulas:

$$m'(f) = \int_G (f \circ \alpha)(x)\ d\mu(x) = \mathrm{mod}_G(\alpha) \int_G f(x)\ d\mu(x) = \mathrm{mod}_G(\alpha)m(f)$$

for all $f \in C_{00}(G)$. The strength of the above statement lies in the fact, that $\mathrm{mod}_G(\alpha)$ only depends on α, so in particular not on m. Furthermore, one readily verifies that the *modular function* $\mathrm{mod}_G : \mathrm{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ is a homomorphism, i.e. $\mathrm{mod}_G(\alpha_1 \circ \alpha_2) = \mathrm{mod}_G(\alpha_1)\mathrm{mod}_G(\alpha_2)\ \forall \alpha_1, \alpha_2 \in \mathrm{Aut}(G)$.

Example 1.12. Here are two examples for non-trivial modular functions:

(i) Let $G = (\mathbb{R}^n, +)$, equipped with the Lebesgue measure. We first have to determine $\mathrm{Aut}(\mathbb{R}^n)$. Clearly, $\mathrm{GL}(n, \mathbb{R}) \leq \mathrm{Aut}(\mathbb{R}^n)$ and in fact equality holds. Indeed, any continuous additive self-map of $\mathbb{R}^n$ is homogeneous and therefore contained in $\mathrm{GL}(n, \mathbb{R})$. On the other hand, there is a wealth of additive, non-continuous, non-homogeneous self-maps of $\mathbb{R}$, namely $\mathrm{GL}(\mathbb{R} / \mathbb{Q})$. These maps are not even measurable. In fact, a consequence of our discussion of the Haar measure will be that a measurable map between locally compact Hausdorff groups is necessarily continuous.

Returning to the originial discussion, one now verifies that the modular function $\mathrm{mod}_G : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\det \alpha|^{-1}$.

(ii) Generalizing the fields $\mathbb{R}$ and $\mathbb{C}$, let k be a locally compact Hausdorff *topological field*. Then in particular, $(k, +)$ and $(k^*, \cdot)$ are locally compact commutative groups which thus have left Haar measures. Every $y \in k^*$ gives rise to the automorphism $\alpha_y \in \mathrm{Aut}((k, +)),\ x \mapsto yx$. Defining

$$|-| : k^* \to \mathbb{R}_{>0},\ y \mapsto \mathrm{mod}_k(\alpha_y),$$

we obtain a multiplicative norm on k^*. One can show that $|y_1 + y_2| \leq c \max\{|y_1|, |y_2|\}$ for some constant $c \in \mathbb{R}_{>0}$. It is further a non-trivial fact that $|-|$ is non-trivial if k is non-discrete. This is the starting point of the classification of all locally compact Hausdorff, non-discrete fields and constitutes a topological way to recover the prime numbers (in form of the fields $\mathbb{Q}_p$)! Also, Fourier analysis and L-functions arise in this context. See for instance [Wei95, Ch. 1].

We now apply the previous discussion to a special class of automorphisms, namely the inner automorphisms of a locally compact Hausdorff group G: Every $g \in G$ gives rise to the automorphism $c_g : G \to G,\ x \mapsto gxg^{-1}$ and we define the *modular function* $\Delta_G : G \to \mathbb{R}_{>0},\ g \mapsto \mathrm{mod}_G(c_g)$. Explicitly, we have for all $g \in G$ and $f \in C_{00}(X)$:

$$\int_G f(xg^{-1})\ d\mu(x) = \int_G f(gxg^{-1})\ d\mu(x) = \Delta_G(g) \int_G f(x)\ d\mu(x).$$

Hence the modular function Δ_G captures the extent to which a left Haar measure fails to be a right Haar measure. Here are two important properties of Δ_G which should be proved as an exercise (see exercise class).

Proposition 1.13. Let G be a locally compact group. Then

(i) $\Delta_G : G \to \mathbb{R}_{>0}$ is a continuous homomorphism, and
(ii) $\forall f \in C_{00}(G) : \int_G f(x^{-1})\Delta_G(x^{-1})\, d\mu(x) = \int_G f(x)\, d\mu(x)$.

In view of the above remark, the following definition suggests itself.

Definition 1.14. A locally compact Hausdorff group G is *unimodular* if and only if $\Delta_G \equiv 1$. Equivalently, G is *unimodular* if every left Haar measure is also a right Haar measure.

Example 1.15. Here are some examples of unimodular and non-unimodular groups.

(i) Any locally compact Hausdorff abelian group is unimodular since left- and right-invariance are equivalent in this case.
(ii) Any discrete group is unimodular, the Haar measure being given by the counting measure.
(iii) Every compact Hausdorff group is unimodular as follows from the defining identity of the modular function, setting $f \equiv 1$ which indeed is a continuous, compactly supported function if G is compact.

Note that any group can be made unimodular by equipping it with the discrete topology. Therefore, unimodularity does not relate to an algebraic property. However, by the third example, it is related to compactness, a topological property.

(iv) Consider the group $G := \mathrm{GL}(n, \mathbb{R}) = \{X \in M_{n,n}(\mathbb{R}) \mid \det X \neq 0\}$ which is a subset of $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n\cdot n}$. In view of this, we write $X = (x_{ij})_{i,j}$ for $X \in \mathrm{GL}(n, \mathbb{R})$. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\cdot n}$, the restriction of the Lebesgue measure $\prod_{i,j=1}^n dx_{i,j}$ to $\mathrm{GL}(n, \mathbb{R})$ provides us with a Radon measure of full support on $\mathrm{GL}(n, \mathbb{R})$. Note that one cannot do this for e.g. $\mathrm{SL}(n, \mathbb{R})$ which is a submanifold of $\mathrm{GL}(n, \mathbb{R})$ of strictly smaller dimension. We define the functional

$$m : C_{00}(\mathrm{GL}(n, \mathbb{R})) \to \mathbb{C},\ f \mapsto \int_G f(X)|\det X|^{-n}\, d\lambda(X)$$

where $\lambda = \prod_{i,j=1}^n dx_{ij}$ is the Lebesgue measure. Note that the above integral is well-defined since $\det X$ stays bounded on the compact support of f. Therefore, m defines a Radon measure on $\mathrm{GL}(n, \mathbb{R})$ which we check to be both left- and right-invariant. In particular, $\mathrm{GL}(n, \mathbb{R})$ is unimodular:

Given $g \in \mathrm{GL}(n, \mathbb{R})$, define $T_g : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ by $X \mapsto gX$. Then $|\det dT_g| \equiv |\det g|^n$ and hence

$$\begin{aligned} m(\lambda_G(g)f) &= \int_G f(g^{-1}X)|\det X|^{-n}\, d\lambda(X) \\ &= |\det g^{-1}|^n \int_G f(g^{-1}X)|\det g^{-1}X|^{-n}\, d\lambda(X) \\ &= |\det g^{-1}|^n \int_G f(X)|\det X|^{-n}|\det dT_{g^{-1}}(X)|^{-1}\, d\lambda(X) \\ &= |\det g^{-1}|^n \int_G f(X)|\det X|^{-n}|\det g^{-1}|^n\, d\lambda(X) \\ &= m(f) \end{aligned}$$

by the change of variables formula. An according computation works for right-invariance.

(v) So far, our examples have all been unimodular groups. Here is a non-unimodular group:

$$P = \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \middle| \begin{array}{c} x \in \mathbb{R}_{>0}, \\ y \in \mathbb{R} \end{array} \right\} \leq \mathrm{GL}(2, \mathbb{R}).$$

One checks that the map

$$m : C_{00}(P) \to \mathbb{C},\ f \mapsto \int_{\mathbb{R}} \int_0^\infty f\left(\begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix}\right) \frac{d\lambda(x)\ d\lambda(y)}{x^2},$$

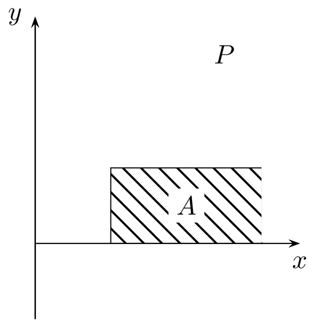
where λ is the Lebesgue measure, defines a left Haar measure on P which is not right-invariant. Indeed, one computes

$$\Delta_P\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}\right) = a^{-2}.$$

Note, that P has infinite measure. However, the subgroup

$$A := \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \middle| \begin{array}{c} x \in \mathbb{R}_{\geq 1}, \\ y \in [0,1] \end{array} \right\}$$

which makes up for a substantial part of the group P has finite measure equal to one as $\int_0^1 \ d\lambda(y) \int_1^\infty \ d\lambda(x)/x^2 = [-1/x]_1^\infty = 1$. Graphically:



Example (v) hints at the utility of parameterization to obtain Haar measures on matrix groups. Another instance of this is the case of $\mathrm{SL}(n, \mathbb{R})$ where the Iwasawa decomposition $\mathrm{SL}(n, \mathbb{R}) = K \cdot A \cdot N$ can be utilized to obtain a measure on $\mathrm{SL}(n, \mathbb{R})$ from the (easy) measures on K, A and N.

We end this section with a characterization of groups of finite Haar measure, underlining again that properties of the Haar measure correspond to topological rather than algebraic properties of the group in question.

Proposition 1.16. Let G be a locally compact Hausdorff group. Then G has finite Haar measure if and only if it is compact.

Proof. If G is compact, then $1 \in C_{00}(G)$ and hence G has finite measure. Conversely, let G be a locally compact Hausdorff group with left Haar measure μ and assume that G is not compact. We aim to prove that G has infinite measure by finding a subset of positive measure which has infinitely many, pairwise disjoint translates in G: Let V be a compact neighbourhood of the identity $e \in G$. Then $\mu(V) > 0$, since V contains a non-empty open set. Then the set $VV^{-1} = \{ab \mid a \in V,\ b \in V^{-1}\}$ is compact as the image of the compact set $V \times V$ under the map $G \times G \to G,\ (x, y) \mapsto xy^{-1}$. Since a finite union of compact sets is compact, we have $\bigcup_{x \in F} xVV^{-1} \neq G$ for every finite set $F \subset G$. We may therefore inductively define a sequence $(x_k)_{k \in \mathbb{N}}$ of points $x_k \in G$ $(k \in \mathbb{N})$ such that $x_n \notin \bigcup_{k=1}^{n-1} x_k VV^{-1}$. Then the sets $x_k V$ are all of positive measure and pairwise disjoint: If $k < n$ and $x_k V \cap x_n V \neq \emptyset$ then there is $v \in V$ such that $x_n \in x_k V v^{-1} \subseteq x_k V V^{-1}$, contradicting the definition of x_n. Therefore, $\mu(G) \geq \mu(\bigcup_{k=1}^{\infty} x_k V) = \sum_{k=1}^{\infty} \mu(x_k V) = \sum_{k=1}^{\infty} \mu(V) = \infty$. □

1.4. **Homogeneous Spaces.** We now extend the theory of Haar measures to the realm of homogeneous spaces. Many spaces on which a group acts are of this extremely versatile type.

First, we remind the reader of some terminology: Let G be a group and let $H \leq G$ be a subgroup of G. A right (left) H-coset is a subset of G of the form xH (Hx) for some $x \in G$. We denote by $G/H = \{xH \mid x \in G\}$ ($H \backslash G$) the corresponding coset space. The set G/H is the set of all equivalence classes of elements of G for the equivalence relation $x \sim y \ :\Leftrightarrow xy^{-1} \in H$, similarly for $H \backslash G$. Denote by $p : G \to G/H$ the canonical projection, i.e. $x \mapsto xH$. Now G acts on the left on G/H via $G \times G/H \to G/H,\ (g, xH) \mapsto (gx)H$ and p is a map of G-spaces, that is $p(g_1 g_2) = g_1 p(g_2)$ for all $g_1, g_2 \in G$. A similar discussion holds of $H \backslash G$. Note that we do not assume H to be a normal subgroup of G.

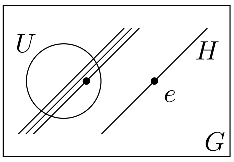
Assume now that G is a topological group and let $H \leq G$ be a subgroup of G. We endow G/H with the quotient topology, i.e. $U \subseteq G/H$ is open if and only if $p^{-1}(U)$ is open in G. This is the finest topology on G/H for which p is continuous. Here are some elementary facts concerning the topology on G/H.

Proposition 1.17. Let G be a topological group and let $H \leq G$ be a subgroup of G. Endow G/H with the quotient topology. Then the following statements hold.

(i) The map $p : G \to G/H$ is open, i.e. the image of an open set under p is open.
(ii) The action $G \times G/H \to G/H,\ (g, xH) \mapsto (gx)H$ is continuous.
(iii) The space G/H is Hausdorff if and only if H is a closed subgroup of G
(iv) If G is locally compact, then G/H is locally compact.
(v) If G is locally compact and $H \leq G$ is closed, then for every compact set $C \subseteq G/H$ there is a compact set $K \subseteq G$ such that $p(K) = C$.

Note that in order to apply measure theory to homogeneous spaces G/H, we need G/H to be locally compact Hausdorff, hence the above proposition.

Proof. (Proposition 1.17). As to (i), let $U \subseteq G$ be open. By definition, $p(U)$ is open if and only if $p^{-1}(p(U))$ is. But $p^{-1}(p(U)) = p^{-1}(\{xU \mid x \in U\}) = UH$ is open as a union $UH = \bigcup_{h \in H} Uh$ of open sets. Graphically, the inverse image of some set under p contains for each of its points all its equivalents; it is saturated under the equivalence relation:



Part (ii) is left as an exercise. It reduces to the axioms of a topological group.

As to part (iii), assume that G/H is Hausdorff. Then points in G/H are closed and hence so is $p^{-1}(eH) = H \leq G$. Conversely, assume that H is closed and let xH and yH be distinct points in G/H. Then $yHx^{-1} \subseteq G$ is closed and does not contain the identity. Hence there is an open neighbourhood $V \subseteq G$ of $e \in G$ such that $V^{-1}V \subseteq G - yHx^{-1}$ (by continuity of the map $G \times G \to G,\ (x, y) \mapsto x^{-1}y$). Then VxH and VyH are disjoint open neighbourhoods of $xH \in G/H$ and $yH \in G/H$.

As to (iv), it suffices to show that every open neighbourhood U of $eH \in G/H$ contains a compact neighbourhood of eH. Since p is continuous and G is locally compact, there is a compact neighbourhood K of $e \in G$ contained in $p^{-1}(U)$; and as p is continuous and open, $p(K)$ is a compact neighbourhood of $eH \in G/H$ contained in U.

For part (v), let V be an open relatively compact neighbourhood of $e \in G$. Then $(p(Vx))_{x\in G}$ is an open cover of C. Since the latter is compact, there is a finite subcover and hence there are $x_1, \ldots, x_n \in G$ such that $C \subseteq p(\bigcup_{i=1}^n Vx_i)$. Then

$$K := \bigcup_{i=1}^{n} \overline{V} x_i \cap p^{-1}(C)$$

is a compact subset of G satisfying $p(K) = C$. Note that K is compact as a closed subset of a compact set and that $p^{-1}(C)$ is closed because p is continuous and G/H is Hausdorff. ☐

Overall, we record that a quotient of a locally compact Hausdorff group by a closed subgroup is again locally compact Hausdorff, the context in which our measure theory will apply.

Example 1.18. Here are examples of homogeneous spaces.

(i) Let $G = \mathbb{R}^n$ and $H = \mathbb{Z}^n$. Then two vectors in G are equivalent if they differ by an integer vector. By Proposition 1.17, the homogeneous space G/H is locally compact Hausdorff and in fact compact as the image of the compact set $[0, 1]^n$ under the canonical projection p. Note that $H = \mathbb{Z}^n$ is a discrete subgroup of $G = \mathbb{R}^n$. This implies that $p : G \to G/H$ is a covering map, meaning that every point $xH \in G/H$ has an open neighbourhood U whose inverse image under p is a disjoint union of subsets of G each of which maps homeomorphically to U under p. For instance, $p(B(0, 1/3))$ is an open neighbourhood of $p(0) = 0 + \mathbb{Z}^n$ such that $p^{-1}(p(B(0, 1/3)))$ is $\bigcup_{\gamma\in\mathbb{Z}^n} \gamma + B(0, 1/3) \cong \mathbb{Z}^n \times B(0, 1/3)$:

Similary, if G is any topological group and $H \leq G$ is a discrete subgroup, then $p : G \to G/H$ is a covering map.

Homogeneous spaces often arise in the context of transitive group actions which we presently recall: An action of a group G on a set X via an action map $G\times X \to X$, $(g, x) \mapsto g_*x$ is *transitive* if and only if for all $x, y \in X$ there is $g \in G$ such that $g_*x = y$. Then for any fixed $x \in X$, the map

$$\varphi_x : G/\mathrm{Stab}_G(x) \to X, \ [g] \mapsto g_*x$$

defines an isomorphism of G-spaces; that is, φ_x is well-defined, bijective and intertwines the left action of G on $G/\mathrm{Stab}_G(x)$ with the action of G on X.

The following examples are instances of this fact and also demonstrate that homogeneous spaces are often useful to put all kinds of structure (algebraic, topological, differential, measure-theoretic, ...) on sets that do not come equipped with such structure in nature.

(ii) Consider the natural action of $G = \mathrm{SO}(n + 1, \mathbb{R})$ on $X = S^n \subset \mathbb{R}^{n+1}$ and define $x = e_{n+1} \in \mathbb{R}^{n+1}$. Then

$$H := \mathrm{Stab}_G(x) = \left\{ \begin{pmatrix} \mathrm{SO}(n, \mathbb{R}) & \\ & 1 \end{pmatrix} \right\} \cong \mathrm{SO}(n, \mathbb{R})$$

and therefore $X = S^n \cong \mathrm{SO}(n + 1, \mathbb{R})/\mathrm{SO}(n, \mathbb{R}) = G/H$ as G-spaces. This bijection is actually a homeomorphism as can be seen by applying the fact, that a continuous bijective map from a compact space into a Hausdorff space is closed and hence a homeomorphism, to the map φ of the following diagram:

$$\begin{tikzcd}
\mathrm{SO}(n+1,\mathbb{R}) \arrow[r, "g\mapsto g_* e_{n+1}"] \arrow[d, "\pi"'] & S^n \\
\mathrm{SO}(n+1,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

This example can also be utilized to prove by induction that $\mathrm{SO}(n,\mathbb{R})$ is connected for all $n \geq 0$. Simply combine the above with the fact that if a subgroup H of a topological group G and the quotient G/H are connected, then so is G.

(iii) Let $1 \leq p \leq n$ and consider $\mathbb{R}^n$ with the Euclidean scalar product. We denote by $\mathrm{ON}(n,p) = \{(f_1,\dots,f_p) \mid \langle f_i, f_j\rangle = \delta_{ij}\}$ the set of orthonormal p-frames in $\mathbb{R}^n$. A priori, $\mathrm{ON}(n,p)$ is just a set. However, we can exploit its symmetries to obtain additional structure: Let $(e_i)_{i=1}^n$ denote the standard basis of $\mathbb{R}^n$. Then the componentwise action of $G = \mathrm{O}(n,\mathbb{R})$ on $\mathrm{ON}(n,p)$ has stabilizer $H = \{g \in \mathrm{O}(n,\mathbb{R}) \mid ge_i = e_i \ \forall i \in \{1,\dots,p\}\} \cong \mathrm{O}(n-p,\mathbb{R})$ of $(e_i)_{i=1}^p \in \mathrm{ON}(n,p)$. Hence

$$\mathrm{ON}(n,p) \cong \mathrm{O}(n,\mathbb{R})/\mathrm{O}(n-p,\mathbb{R})$$

is a compact Hausdorff continuous G-space and it will therefore make sense to talk about nearness of frames and measures of sets of frames.

(iv) Consider the upper half plane $\mathbb{H} = \{z = x+iy \in \mathbb{C}^2 \mid x,y \in \mathbb{R},\ y>0\}$ on which $G = \mathrm{SL}(2,\mathbb{R})$ acts transitively by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_* z := \frac{az+b}{cz+d}$$

To see transivitity, note that for $x \in \mathbb{R}$ and $y \in \mathbb{R} > 0$,

$$\begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y}^{-1} \end{pmatrix}_* i = x+iy.$$

One readily checks $\mathrm{Stab}_G(i) = \mathrm{SO}(2,\mathbb{R})$ whence $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ as G-spaces. Again, this is actually a homeomorphism as can be checked directly by writing down an inverse for the map $\mathrm{SL}(2,\mathbb{R}) \to \mathbb{H},\ g \mapsto g_* i$ using the above transitivity proof.

(v) Related to the preceding example is the following: Let $X := \mathrm{Sym}_1^+(n)$ be the space of symmetric positive definite matrices of determinant one. Then $G := \mathrm{SL}(n,\mathbb{R})$ acts transitively on X via $g_* A := g^T A g$ ($g \in G,\ A \in \mathrm{Sym}_1^+(n)$. Note that this amounts to changing basis for the bilinear form A. The stabilizer of $\mathrm{Id}_n \in \mathrm{Sym}_1^+(n)$ is given by

$$\mathrm{Stab}_G(\mathrm{Id}_n) = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid g^T g = \mathrm{Id}_n\} = \mathrm{SO}(n,\mathbb{R})$$

Hence $\mathrm{Sym}_1^+(n) \cong \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. Combining this with the preceding example for $n=2$, we obtain

$$\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathrm{Sym}_1^+(n)$$

where the G-space isomorphisms are actually G-homeomorphisms. Once we get to the theory of Lie groups, we will see that these maps are actually G-diffeomorphisms. It is therefore possible to carry over the Riemannian metric that $\mathbb{H}$ may be equipped with to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ and $\mathrm{Sym}_1^+(n)$ which may then serve as models for two-dimensional hyperbolic space just as the upper half plane, the hyperboloid, the Poincaré disk or the Klein disk model.

(vi) Let $G = \mathrm{SL}(n, \mathbb{R})$ act on $\mathbb{P}^{n-1}\mathbb{R} = \{V \leq \mathbb{R}^n \mid \dim V = 1\}$ by $g_* V = gV$ ($g \in G,\ V \in \mathbb{P}^{n-1}\mathbb{R}$). Then the stabilizer of $\langle e_1 \rangle$ where e_1 denotes the first standard basis vector is given by

$$P := \mathrm{Stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} a & x \\ & A \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{R} - \{0\},\, x \in \mathbb{R}^{(n-1)\times 1}, \\ A \in \mathrm{GL}(n-1, \mathbb{R}),\, a \det A = 1 \end{array} \right\}$$

whence $\mathbb{P}^{n-1}\mathbb{R} \cong \mathrm{SL}(n, \mathbb{R})/P$. Again, this map is a G-homeomorphism. Note that $\mathbb{P}^{n-1}\mathbb{R}$ is compact. And yet we will see later, however, that there is no $SL(2, \mathbb{R})$-invariant Radon measure on $\mathbb{P}^1\mathbb{R}$.

(vii) We conclude this list of examples with one that is intensely being studied in various parts of mathematics. Let

$$L := \{\mathbb{Z} f_1 + \cdots + \mathbb{Z} f_n \mid f_i \in \mathbb{R}^n \ \ \forall i \in \{1, \ldots, n\},\ \det(f_1 \cdots f_n) = 1\},$$

the space of lattices of covolume one in $\mathbb{R}^n$. The group $G = \mathrm{SL}(n, \mathbb{R})$ clearly acts transitively on L via $g_*(\mathbb{Z} f_1 + \cdots + \mathbb{Z} f_n) := \mathbb{Z}\, g f_1 + \cdots + \mathbb{Z}\, g f_n$. The stabilizer of $\mathbb{Z}\, e_1 + \cdots + \mathbb{Z}\, e_n$ is given by $\mathrm{SL}(n, \mathbb{Z})$, hence $L \cong \mathrm{SL}(n, \mathbb{R}) / \mathrm{SL}(n, \mathbb{Z})$. Therefore, L acquires the structure of a locally compact Hausdorff space on which $\mathrm{SL}(n, \mathbb{R})$ acts continuously.

In number theory, $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SL}(n, \mathbb{Z})$ is also viewed as a space of equivalence classes of binary quadratic forms. It further constitutes a key example in (homogeneous) dynamics.

Now, let G be a locally compact Hausdorff group and let H be a closed subgroup of G. Then G/H is again locally compact Hausdorff and we ask the question under which circumstances there is a G-invariant Radon measure on G/H. Clearly, if H is normal, then G/H is a locally compact Hausdorff group and hence its left Haar measure serves. So the question really is, what happens if H is not normal?

Example 1.19. Here are the two possibilities.

(i) Consider the natural action of $G = \mathrm{SL}(2, \mathbb{R})$ on $X = \mathbb{R}^2 - \{0\}$. Then

$$H := \mathrm{Stab}_G((1, 0)^T) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$$

and hence $G/H \cong X$ has a G-invariant measure, namely the restricted two-dimensional Lebesgue measure.

(ii) On the other hand, G acts on $X = \mathbb{P}^1\mathbb{R} = \{V \leq \mathbb{R}^2 \mid \dim V = 1\}$. Here,

$$H := \mathrm{Stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R} - \{0\},\ x \in \mathbb{R} \right\}$$

and $G/H \cong X$ does not have a G-invariant Radon measure as can be proven directly.

The reader is encouraged to convince himself that these two examples are in accordance with the following general theorem.

Theorem 1.20. Let G be a locally compact Hausdorff group with left Haar measure μ_G and let $H \leq G$ be a closed subgroup of G with left Haar measure μ_H. Then there exists a G-invariant Radon measure $\mu_{G/H}$ on G/H if and only if $\Delta_G|_H \equiv \Delta_H$. In this case, $\mu_{G/H}$ is unique up to strictly positive scalar multiples and suitably normalized satisfies for all $f \in C_{00}(G)$:

$$\int_G f(g)\ d\mu_G(g) = \int_{G/H} \int_H f(gh)\ d\mu_H(h)\ d\mu_{G/H}(gH).$$

Roughly, the above integration formula means that integration over G is equivalent to summing up the integrals over all cosets, which form a partition of G, computed using the Haar measure on H. It is called the Weil integration formula.

Proof. The proof consists of three parts. The first part is a description of $C_{00}(G/H)$ since a measure on G/H is going to be constructed via a functional on $C_{00}(G/H)$:

Lemma. The following map is surjective.

$$T_H : C_{00}(G) \to C_{00}(G/H),\ f \mapsto \left(gH \mapsto \int_H f(gh)\ d\mu_H(h) \right)$$

Proof. In words, T_H associates to a function $f \in C_{00}(G)$ the function on G/H which on a particular coset has as value the integral of f over that coset, computed using the Haar measure on H.

Now, several things need to be checked. First of all, for all $f \in C_{00}(G)$ and for all $gH \in G/H$, the integral $\int_H f(gh)\ d\mu_H(h)$ is independent of the representative of gH and finite. Next, for all $f \in C_{00}(G)$, the function $T_H f$ is continuous as a parametrized integral as in the proof of the continuity of the modular function. Clearly, $\operatorname{supp} T_H f \subseteq p(\operatorname{supp}(f))$ and hence $T_H f \in C_{00}(G/H)$. It remains to prove that T_H is surjective. To this end, let $F \in C_{00}(G/H)$. Pick $K \subseteq G$ such that $p(K) = \operatorname{supp} F$ (Proposition 1.17) and let $\eta \in C_{00}(G)$ satisfying $K \prec \eta$ (Urysohn's Lemma). Now define $f \in C_{00}(G)$ by

$$f : G \to \mathbb{C},\ g \mapsto \begin{cases} \frac{((F \circ p)\cdot\eta)(g)}{T_H \eta \circ p(g)} & T_H\eta \circ p(g) \neq 0 \\ 0 & T_H \eta \circ p(g) = 0 \end{cases}$$

Again, we need to show that this function is continuous and has compact support. As for compact support, clearly $\operatorname{supp} f \subseteq \operatorname{supp} \eta$. In fact, if G was compact, we could choose $\eta \equiv 1$. To show that f is continuous, we show that it is continuous on two open sets $U_1 \subseteq G$ and $U_2 \subseteq G$ satisfying $U_1 \cup U_2 = G$. On the set $U_1 := \{g \in G \mid T_H \eta \circ p(g) \neq 0\}$ it is continuous as a quotient of continuous functions; and on $U_2 := G - KH$ it is continuous as it vanishes there. Further, if $g \notin U_1$, then $0 = T_H \eta \circ p(g) = \int_H \eta(gh)\ d\mu_H(h)$. Since η is a non-negative continuous function, this implies $\eta(gh) = 0$ for all $h \in H$, hence $g \notin KH$, i.e. $g \in U_2$. Continuity and compact support being established, it remains to show that $T_H f \equiv F$. Compute

$$T_H f(gH) = \int_H \frac{F(ghH)\eta(gh)}{T_H \eta(ghH)}\ d\mu_H(h) = F(gH) \frac{\int_H \eta(gh)\ d\mu_H(h)}{T_H \eta(gH)} = F(gH).$$

Hence T_H is surjective. ☐

We now proceed by showing that the condition $\Delta_G|_H \equiv \Delta_H$ is necessary. This does not use the above lemma. Assume that a measure $\mu_{G/H}$ as stated exists. Then the functional $m : C_{00}(G) \to \mathbb{C}$,

$$f \mapsto \int_{G/H} T_H f(gH)\ d\mu_{G/H}(gH) = \int_{G/H} \int_H f(gh)\ d\mu_H(h)\ d\mu_{G/H}(gH)$$

defines a left Haar measure on G. In particular $m(\varrho(t^{-1})f) = \Delta_G(t) m(f)$ for all $f \in C_{00}(G)$ and $t \in G$. On the other hand, we have for all $t \in H$:

$$m(\varrho(t^{-1})f) = \int_{G/H} \int_H (\varrho(t^{-1})f)(gh)\ d\mu_H(h)\ d\mu_{G/H}(gH) = \Delta_H(t) m(f).$$

Choosing $f \in C_{00}(G)$ such that $m(f) \neq 0$, we conclude that $\Delta_G|_H \equiv \Delta_H$.

It remains to prove that the condition $\Delta_G|_H \equiv \Delta_H$ is sufficient for the existence of $\mu_{G/H}$. We would like to define a functional on $C_{00}(G/H)$ by

$$m : C_{00}(G/H) \to \mathbb{C},\ F \mapsto \int_G f(g)\ d\mu_G(g),$$

where f is chosen such that $T_H f \equiv F$. Once we have shown that m is well-defined, it is a G-invariant positive linear functional on G/H and hence defines a measure $\mu_{G/H}$ as asserted. To show that $\int_G f(g)\ d\mu(g)$ does not depend on the choice of $f \in T_H^{-1}(F)$, it suffices to show that $\int_G f(g)\ d\mu_G(g) = 0$ whenever $\int_H f(gh)\ d\mu_H(h) = 0$ for all $g \in G$.

To this end, we note that $\Delta_G|_H \equiv \Delta_H$ implies that for all $f_1, f_2 \in C_{00}(G)$:

$$\int_G f_1(g) \int_H f_2(gh)\ d\mu_H(h)\ d\mu_G(g) = \int_G f_2(g) \int_H f_1(gh)\ d\mu_H(h)\ d\mu_G(g).$$

Indeed, we compute

$$\begin{aligned}\int_G f_1(g) \int_H f_2(gh)\ d\mu_H(h)\ d\mu_G(g) &= \int_H \int_G f_1(g) f_2(gh)\ d\mu_G(g)\ d\mu_H(h)\\ &= \int_H \int_G f_1(g'h^{-1}) f_2(g')\ \Delta_G(h^{-1})\ d\mu_G(g')\ d\mu_H(h)\\ &= \int_G \int_H f_2(g') f_1(g'h^{-1}) \Delta_H(h^{-1})\ d\mu_H(h)\ d\mu_G(g')\\ &= \int_G f_2(g') \int_H f_1(g'h')\ d\mu_H(h')\ d\mu_G(g').\end{aligned}$$

Applying this to $f_1 := f$ and choosing $f_2 \in C_{00}(G)$ such that $\int_H f_2(gh)\ d\mu_H(h) = 1$ for all $g \in \operatorname{supp} f_1$ (bysurjectivity of T_H) proves that $\int_G f(g)\ d\mu_G(g) = 0$ in case $\int_H f(gh)\ d\mu_H(h) = 0$ for all $g \in G$.

Hence, the functional m provides a measure $\mu_{G/H}$ on G/H as asserted which satisfies the integration formula since

$$\begin{aligned}\int_G f(g)\ d\mu_G(g) = m(F) &= \int_{G/H} F(gH)\ d\mu_{G/H}(gH)\\ &= \int_{G/H} T_H f(gH)\ d\mu_{G/H}(gH)\\ &= \int_{G/H} \int_H f(gh)\ d\mu_H(h)\ d\mu_G(g).\end{aligned}$$

Any other G-invariant Radon measure on G/H is readily seen to be a multiple of the one constructed. □

Remark. The reader is encouraged to go through the above proof to check that if G is *compact*, the produced quotient measure $\mu_{G/H}$ on G/H is given for every measurable set $E \subseteq G/H$ by

$$\mu_{G/H}(E) = \frac{\mu_G(p^{-1}(E))}{\mu_H(H)};\ \text{in particular}\ \mu_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)}.$$

This maybe deserves to be called a quotient measure.

To illustrate the usefulness of Theorem 1.20, we use the theory of homogeneous spaces to explicitly construct a Haar measure on $\mathrm{SL}(2,\mathbb{R})$. To this end, recall that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on the upper half plane $\mathbb{H}$ with stabilizer of $i \in \mathbb{H}$ being $\mathrm{SO}(2,\mathbb{R})$, and that the canonical map $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathbb{H}$ is a homeomorphism.

Now, note that $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(2,\mathbb{R})$ are unimodular: $\mathrm{SO}(2,\mathbb{R})$ is unimodular because it is compact and $\mathrm{SL}(2,\mathbb{R})$ is unimodular because it equals its commutator subgroup; for instance, one computes for $a \in \mathbb{R} - \{0\}$ and $x \in \mathbb{R}$:

$$\left[\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right] = \begin{pmatrix} 1 & (a^2-1)x \\ & 1\end{pmatrix}.$$

Then recall that by Gauss elimination any matrix in $\mathrm{SL}(2,\mathbb{R})$ is a product of such matrices and the lower-triangular matrices computed analogously. (A similar argument works for $\mathrm{SL}(n,\mathbb{R})$ using various inclusions of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$ and in fact for all coefficient fields containing an element $a \neq 0$ such that $a^2 \neq 1$.).

Now, by Theorem 1.20, there is an $\mathrm{SL}(2,\mathbb{R})$-invariant Radon measure on $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$. Actually, one verifies, that $d\mu(x,y) = d\lambda(x)\ d\lambda(y)/y^2$ is such a measure on $\mathbb{H}$. It comes from the $\mathrm{SL}(2,\mathbb{R})$-invariant hyperbolic metric. Further, using the homeomorphism $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ it follows from Theorem 1.20 that the functional $C_{00}(\mathrm{SL}(2,\mathbb{R})) \to \mathbb{C}$,

$$f \mapsto \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} f\left(\begin{pmatrix}\sqrt{y} & x\sqrt{y}^{-1}\\ & \sqrt{y}^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right)\, d\theta\, \frac{d\lambda(y)}{y^2}\, d\lambda(x)$$

defines a Haar measure on $\mathrm{SL}(2,\mathbb{R})$.

1.5. **Continuity of measurable homomorphisms.** In this section, we prove the following theorem which relies solely on the existence of the Haar measure and its properties.

Theorem 1.21 (Mackey, 1950's). Let G and H be locally compact Hausdorff, second countable groups. If $\varphi : G \to H$ is a measurable homomorphism, then φ is continuous.

In the proof, we will make use of the following two lemmas which are of independet interest. One is purely topological, the other mostly measure theoretic.

Lemma 1.22. Let G be a topological group. Further, let $A \subseteq G$ be compact and $W \subseteq G$ be open such that $A \subseteq W \subseteq G$. Then there is a neighbourhood $N \subseteq G$ of the identity $e \in G$ such that $NA \subseteq W$.

Proof. Since W is open, there is for each $x \in A \subseteq W$ a neighbourhood $U_x \in \mathcal{U}(e)$ such that $U_x x \subseteq W$. In particular, $A \subseteq \bigcup_{x\in A} U_x x \subseteq W$. Compactness of A now implies that there are finitely many $x_1, \dots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^n U_{x_i} x_i \subseteq W$. Then $N := \bigcap_{i=1}^n U_{x_i}$ serves: Indeed,

$$NA \subseteq N\bigcup_{i=1}^n U_{x_i}x_i \subseteq \bigcup_{i=1}^n NU_{x_i}x_i \subseteq \bigcup_{i=1}^n U_{x_i}^2 x_i \subseteq W.$$

$\square$

Lemma 1.23. Let G be a locally compact Hausdorff group and let μ be a Haar measure on G. Further, let $E \subseteq G$ be measurable with $\mu(E) > 0$. Then EE^{-1} is a neighbourhood of the identity.

This result is astonishing in the case $G = (\mathbb{R}, +)$ already: Think of E being a Cantor set of positive measure, not containing any open set. The proof uses all the regularity properties of the Haar measure.

Proof. By inner regularity, there is a compact set $A \subseteq E$ with $\mu(A) > 0$. Then $EE^{-1} \supseteq AA^{-1}$ and hence it suffices to show that AA^{-1} contains a neighbourhood of the identity. Since A is compact, it has finite measure and hence by outer regularity, there is an open set $W \supseteq A$ such that $\mu(A) \le \mu(W) < 2\mu(A)$. Now, by Lemma 1.22 there is a neighbourhood $N \in \mathcal{U}(e)$ such that $NA \subseteq W$. The condition $\mu(W) < 2\mu(A)$ implies that $nA \cap A \neq \emptyset$ for all $n \in N$. Hence $N \subseteq AA^{-1} \subseteq EE^{-1}$ as claimed. $\square$

We are now in a position to prove Theorem 1.21.

Proof. (Theorem 1.21). It suffices to show that φ is continuous at the identity. Let $U \in \mathcal{U}(e)$ be an open neighbourhood of $e \in H$. We aim to show that $\varphi^{-1}(U)$ contains an open set. Pick $V \in \mathcal{U}(e)$ such that $VV^{-1} \subseteq U$. Since H is second countable, so is $\varphi(G)$. Hence there exists a countable dense subset $(h_n)_{n\in\mathbb{N}}$ of $\varphi(G)$ satisfying $\varphi(G) \subseteq \bigcup_{n\in\mathbb{N}} h_n V$. Then there are $(g_n)_{n\in\mathbb{N}}$ such that $\varphi(g_n) = h_n$ and

$G = \bigcup_{n\in\mathbb{N}} g_n\varphi^{-1}(V)$. In particular, there is an $n_0 \in \mathbb{N}$ such that $\mu(g_{n_0}\varphi^{-1}(V)) > 0$. Hence also $\mu(\varphi^{-1}(V)) > 0$ since μ is left-invariant. By Lemma 1.23 we thus deduce, using that φ is a homomorphism, that

$$\varphi^{-1}(V)\varphi^{-1}(V)^{-1} \subseteq \varphi^{-1}(VV^{-1}) \subseteq \varphi^{-1}(U)$$

contains an open neighbourhood of the identity $e \in G$. $\square$

As an example we conclude that every measurable homomorphism φ from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ is of the form $x \mapsto \alpha x$ for some $\alpha \in \mathbb{R}$: By Theorem 1.21, φ is continuous; and a continuous, additive map from $\mathbb{R}$ to $\mathbb{R}$ is homogeneous whence linear; thus the assertion.

2. Lie Groups

Having dealt with topological groups for some time, we are now well-prepared to start the discussion of Lie groups. First of all, we give the definition and examples.

2.1. **Lie Groups and Examples.**

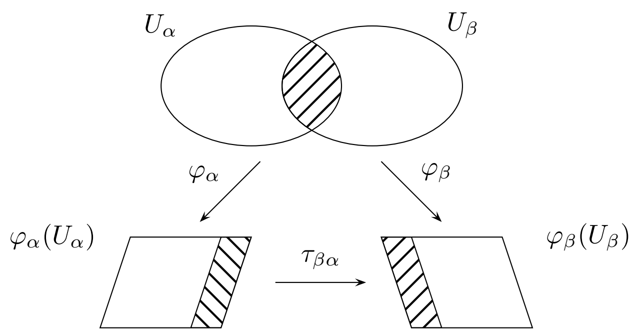
Definition 2.1. A *Lie group* is a group G endowed with the structure of a smooth manifold such that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth maps.

By the underlying definitions, which we shall recall presently, a Lie group is in particular a locally compact Hausdorff second countable topological group.

Definition 2.2. A *topological n-manifold* is a Hausdorff, second countable topological space such that every point has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$. A pair (U, φ) consisting of an open subset $U \subseteq M$ and a map $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, which is a homeomorphism onto its image, is a *(coordinate) chart* at any point of U. A *smooth structure* on M is an *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts whose domains cover M and such that for all $\alpha, \beta \in A$, the map

$$\tau_{\beta\alpha} := \left(\varphi_\beta|_{U_\alpha\cap U_\beta}^{\varphi_\beta(U_\alpha\cap U_\beta)}\right) \circ \left(\varphi_\alpha|_{U_\alpha\cap U_\beta}^{\varphi_\alpha(U_\alpha\cap U_\beta)}\right)^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth. Here is a picture of the above situation:



In this case, M is a *smooth n-manifold.*

We may then for instance say, that a function $f : M \to \mathbb{R}$ is differentiable at $p \in M$ if for some chart (U, φ) at p, the map $f \circ \varphi^{-1} : \varphi(U) \to U \to \mathbb{R}$ is differentiable at $\varphi(p) \in \mathbb{R}^n$. This definition of differentiability does not depend on the chosen chart, due to the smoothness of the transition maps.

Here are some problems associated with the above definitions:

(i) The n in "n-manifold": Could it be that a set M is an n_i-manifold for $n_1 \neq n_2$? In other words, given homeomorphic open sets $U_1 \subseteq \mathbb{R}^{n_1}$ and $U_2 \subseteq \mathbb{R}^{n_2}$, is it true that $n_1 = n_2$? This is easy to show using the linear algebra of derivatives if U_1 and U_2 are diffeomorphic. It remains true in the topological case but known proofs use for instance singular homology; a tool which wasn't available at the time Riemann defined topological n-manifolds.

(ii) Existence of smooth structures: Are there topological n-manifolds that do not admit a smooth structure? This is indeed the case. The first example of such a topological manifold was given by Kervaire in 1960; it uses the root system of the Lie algebra E_8 in a skillful way.

(iii) Uniqueness of smooth structures: Are there inequivalent smooth structures on a given topological n-manifold? For $\mathbb{R}^n$, viewed as a topological manifold, there is a unique smooth structure in all dimensions except for $n = 4$ in which case there are uncountably many. It is known that for a compact topological manifold there is a unique smooth structure in dimensions 1, 2 and 3 and finitely many in any dimension except 4. The first example of inequivalent smooth structures was given by Milnor in 1956 on S^7, termed exotic spheres.

We see that the most simple questions about (topological) manifolds are already impossibly difficult. Hilbert's fifth problem asked whether such "diseases" can also happen to topological groups. In a nutshell, the amazing 1953 answer worked out by many mathematicians including Gleason, Montgomery, Zippin and Yamabe is: A topological group which is a topological manifold can be turned into a Lie group in a unique way. As striking as the result is, its first application had to wait until the 1980's when Gromov applied it for the proof of his polynomial growth theorem.

A beautiful characterization of Lie groups in terms of algebra is the following: A topological group is a Lie group if and only if it has no small subgroups; here, a topological group is said to have small subgroups if every neighbourhood of the identity contains a non-trivial subgroup.

Example. Consider for instance the following: Let $K := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Then K is homeomorphic to a Cantor set as a topological space and hence cannot be a Lie group, not even locally. And in fact, the open subgroups

$$U_n := \underbrace{\{e\} \times \cdots \times \{e\}}_{n} \times \prod_{k=n+1}^{\infty} \mathbb{Z}/2\mathbb{Z} \leq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

form a decreasing neighbourhood basis of the identity.

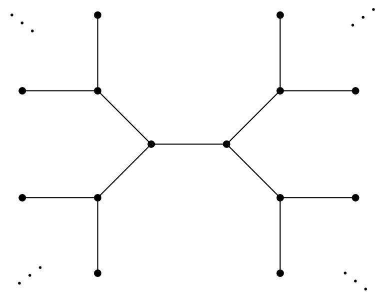
Example 2.3. Let us now check whether the topological groups of example 1.3 can be turned into Lie groups:

(i) A discrete group is a Lie group if and only if it is countable and then has dimension 0. (If it is not countable, it is not second-countable as a topological space.)
(ii) The group $(\mathbb{R}^n, +)$ is clearly a Lie group; addition and taking the negative are smooth maps.

For the next examples, recall that any open subset of a smooth manifold is a smooth manifold with the induced structure.

(iii) The groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ are Lie groups: The underlying sets are open subsets of the smooth manifolds $\mathbb{R}$ and $\mathbb{C}$ respectively. Clearly, multiplication and inversion are smooth with respect to the induced structure.
(iv) The group $\mathrm{GL}(n, \mathbb{R})$ is a smooth n^2-manifold as an open subset of $\mathbb{R}^{n \cdot n}$. The product is a polynomial map, the inversion a rational one, hence both are smooth.
(v) $\mathbb{Q}_p$ is not a Lie group.
(vi) The homeomorphism group $\mathrm{Homeo}(X)$ of a topological space X is in general not even locally compact and hence "too big" to be a Lie group.
(vii) A finite cartesian product of Lie groups with the product smooth structure is again a Lie group.
(viii) If (X, d) is a metric space then $\mathrm{Iso}(X)$ is a locally compact Haudorff topological group which may or may not be a Lie group. For instance, in the case $(X, d) = (\mathbb{R}^n, d_{\mathrm{eucl}})$ we have $\mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, which is a Lie group. More generally, by Steenrod-Myers [MS39], the isometry group of a Rieminnian manifold is a Lie group.

On the other hand, the isometry group of a regular tree $\mathcal{T}_d$, $d \geq 3$ equipped with the combinatorial distance is not a Lie group. One can for instance show that it has small subgroups: A neighbourhood basis of the identity is given by those sets of isometries that fix larger and larger balls pointwise. Clearly, these sets are also non-trivial subgroups.



(x) The group $\mathbb{Z}_p$ is not a Lie group as it is totally disconnected and uncountable.

(xi) Examples xi and xii deal with subgroups of $\mathbb{R}^n$. The subgroup

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \middle| \; \forall\, 1 \leq i \leq n : \; \lambda_i \in \mathbb{R} - \{0\} \right\}$$

is homeomorphic to $\mathbb{R}^{*n}$ and hence can be given the structure of a smooth n-manifold. The group operations in A are indeed smooth with respect to this structure. Therefore, A is a Lie group. The subgroup

$$N = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & 1 \end{pmatrix} \middle| \; \begin{array}{l} \forall\, 1 \leq i \leq n : \; x_{ii} = 1, \\ \forall\, 1 \leq j < i \leq n : \; x_{ij} = 0 \end{array} \right\}$$

is homeomorphic to $\mathbb{R}^{(n-1)/2}$ and hence can be given the structure of a smooth $(n-1)/2$-manifold. Here, multiplication and inversion are polynomials and hence smooth with respect to this structure. Therefore, N is a Lie group.

In order to deal with the subgroups $K = \mathrm{O}(n, \mathbb{R})$ and $\mathrm{O}(p, q)$, we introduce the concept of a regular submanifold.

Definition 2.4. Let M be a smooth m-manifold. A *regular n-submanifold* of M is a topological subspace $N \subseteq M$ such that $\forall p \in N$ there is a chart (U, φ) of M at p such that

(i) $\varphi(p) = 0$,
(ii) $\varphi(U) = (-1, 1)^m$, and
(iii) $\varphi(N \cap U) = \{x \in (-1, 1)^m \mid x_{n+1} = \cdots = x_m = 0\}$.

Observe that a regular submanifold N of a smooth manifold M is a smooth manifold in its own right by restricting the charts of Definition 2.4 to N. The usefulness of regular submanifolds for us lies in the following theorem.

Theorem 2.5. Let G be a Lie group and let $H \leq G$ be a subgroup which is also regular submanifold. Then H is a Lie group.

The proof of 2.5 boils down to the fact that the restriction to a coordinate plane of a smooth map on $\mathbb{R}^n$ is again smooth. In order to show that a given subgroup is also a regular submanifold, or to produce regular submanifolds, a powerful tool is given by the implicit function theorem.

Theorem 2.6. Let M, M' be smooth manifolds of dimension m and m'. Further, let $f : M \to M'$ be a smooth map. Assume that f has constant rank on M. Then for every $q \in f(M)$, the set $f^{-1}(q) \subseteq M$ is a regular submanifold of dimension $m - \mathrm{rank} f$.

Retain the notation of Theorem 2.6. Recall that the rank of f at $p \in M$ is the rank of the linear map $D_p f : T_p M' \to T_{\varphi(p)} M$, i.e. the dimension of its image.

The rank assumption in Theorem 2.6 is totally essential. Without it, pretty much everything can happen: For instance, every closed subset $F \subseteq \mathbb{R}^n$ is the zero set of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

Typically, a function f will have constant rank on an open subset which we then define to be M. In the following, we will deal with the open subset $\mathrm{GL}(n, \mathbb{R})$ of $\mathbb{R}^{n \cdot n}$.

Example 2.7. As an illustration, we consider the subgroups $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n, \mathbb{R})$ and $\mathrm{O}(p, q)$ for $p + q = n$ of $\mathrm{GL}(n, \mathbb{R})$.

(i) $\mathrm{SL}(n, \mathbb{R}) \leq \mathrm{GL}(n, \mathbb{R})$ is a regular $n^2 - 1$-submanifold of $\mathrm{GL}(n, \mathbb{R})$ and hence a Lie group by Theorem 2.5: Consider the smooth determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. In order to show that it has constant (and maximal) rank 1, it suffices to show that $D_A \det$ does not vanish for any $A \in \mathrm{GL}(n, \mathbb{R})$. For $X \in M_{n,n}(\mathbb{R}) = T_A \, \mathrm{GL}(n, \mathbb{R})$, we compute

$$\begin{aligned} D_A \det(X) &= \left.\frac{d}{dt}\right|_{t=0} \det(A + tX) \\ &= \det A \left.\frac{d}{dt}\right|_{t=0} \det(I + tA^{-1}X) = \det A D_I \det(A^{-1}X) \end{aligned}$$

Since the map $M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ X \mapsto A^{-1}X$ is an isomorphism, the above implies that $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ has constant rank. We continue by computing $D_I \det(X)$:

$$\begin{aligned} D_I \det(X) &= \left.\frac{d}{dt}\right|_{t=0} \det(I + tX) = \left.\frac{d}{dt}\right|_{t=0} t^n \det(t^{-1}I + X) \\ &= \left.\frac{d}{dt}\right|_{t=0} t^n \det(t^{-1}I - (-X)) = \left.\frac{d}{dt}\right|_{t=0} t^n C_{-X}(t^{-1}) \end{aligned}$$

where $C_Y(t) = \det(tI - Y)$ is the characteristic polynomial of a matrix $Y \in M_{n,n}(\mathbb{R})$. Hence the above equals

$$= \left.\frac{d}{dt}\right|_{t=0} t^n \left((t^{-1})^n - (t^{-1})^{n-1} \operatorname{tr}(-X) + (t^{-1})^{n-2} P(t)\right)$$

where $P(t) \in \mathbb{R}[t]$ is a polynomial in t. Consequently, we obtain

$$D_I \det(X) = \left.\frac{d}{dt}\right|_{t=0} \det(I + tX) = \operatorname{tr}(X)$$

and hence $D_I \det$ has rank 1; and so does $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ by the above reasoning.

(ii) $\mathrm{O}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = \mathrm{Id}_n\} \leq \mathrm{GL}(n, \mathbb{R})$ is a regular $n(n-1)/2$-submanifold of $\mathrm{GL}(n, \mathbb{R})$ and hence is a Lie group by Theorem 2.5: Consider the smooth map $f : \mathrm{GL}(n, \mathbb{R}) \to M_{n,n}(\mathbb{R}),\ A \mapsto A^T A$. Again, we compute for $X \in M_{n,n}(\mathbb{R}) = T_A \operatorname{GL}(n, \mathbb{R})$

$$\begin{aligned} D_A f(X) &= \frac{d}{dt}\Big|_{t=0} (A + tX)^T (A + tX) \\ &= \frac{d}{dt}\Big|_{t=0} (A^T A + tA^T X + tX^T A + t^2 X^T X) \\ &= A^T X + X^T A \end{aligned}$$

In particular, $D_A f(A^{-1T} X) = X + X^T = D_I f(X)$ which as above shows that f has constant rank. To compute the rank, note that

$$D_I f : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ X \mapsto X + X^T$$

has image exactly the subspace of $M_{n,n}(\mathbb{R})$ consisting of all symmetric matrices. Since a symmetric matrix is determined by its upper triangle, the dimension of this space is $1+2+\cdots+n = n(n+1)/2$. As a result, $\mathrm{O}(n, \mathbb{R})$ is a regular submanifold of $\operatorname{GL}(n, \mathbb{R})$ of dimension $n^2 - n(n+1)/2 = n(n-1)/2$; for instance $\dim \mathrm{O}(3) = 3$.

(iii) The case of $\mathrm{O}(p, q) \leq \operatorname{GL}(n, \mathbb{R})$ is now left as an exercise.

Summing up the above and what is to come, one of the main difficulties in grasping Lie theory lies in the fact that spaces of matrices are vector spaces themselves on which we can consider linear maps.

One of the theorems we will prove and which constitutes a shortcut to the above three examples is the following: Every closed subgroup of a Lie group is a Lie group in a unique way.

2.2. **Vector Fields and Lie Algebras.** In this section, we will see that the tangent space at the identity of a Lie group remembers some of the group structure of G, namely, in a sense, it linearizes the group law of G. It will be called the Lie algebra of G. To explain this, we recall or rather renew the definition of a tangent space: Let M be a smooth manifold and let $p \in M$. Recall that $C^\infty(p)$ denotes the space of germs of C^∞-functions at p: It will capture that a derivative of a function at a point can be computed given an arbitrarily small neighbourhood of that point. Let

$$F(p) := \{(U, f) \mid U \in \mathcal{U}(p) \text{ open},\ f : U \to \mathbb{R} \text{ smooth}\}$$

and introduce on $F(p)$ the equivalence relation $\sim$ given by

$$(U_1, f_1) \sim (U_2, f_2) \ \Leftrightarrow\ \exists\, U_3 \in \mathcal{U}(p) \text{ open},\ U_3 \subseteq U_1 \cap U_2 :\ f_1|_{U_3} \equiv f_2|_{U_3}$$

Then $C^\infty(p) := F(p)/\sim$. An element of $C^\infty(p)$ has a well-defined value at $p \in M$.

Definition 2.8. Let M be a smooth manifold and let $p \in M$. A *tangent vector* of M at p is a linear form $X_p : C^\infty(p) \to \mathbb{R}$ which satisfies Leibniz' rule: For all $f, g \in C^\infty(p)$ we have $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$. The set of tangent vectors of M at p is denoted T_pM.

This definition of a tangent vector may seem unfamiliar but has many advantages to offer. Of course, it yields the same object as other definitions you may have seen: Clearly, it is a vector space. Further, if (U, φ) is a chart of M at p with $\varphi(p) = 0$, then the map

$$\mathbb{R}^n \to T_p(M),\ v \mapsto \big(X_p^v : f \mapsto D_0(f \circ \varphi^{-1})(v)\big)$$

is a vector space isomorphism. Here the right hand expression could be called the derivative of f at p in the direction v. Proving this requires some extra knowledge on T_pM as defined above and a skillful manipulation of Taylor expansions, see e.g. [War71].

Now, let $TM := \bigcup_{p\in M} T_pM$ be the *tangent bundle* of M. One can endow TM with the structure of a smooth manifold in such a way that $\pi : TM \to M$ is smooth. Then a *smooth vector field* on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. Thus, a vector field on M is a function which in a smooth fashion to every point associates a tangent vector at that point. We will work with the following precise definition.

Definition 2.9. Let M be a smooth manifold. A *vector field* on M is a map $X : M \to TM,\ p \mapsto X_p$ such that $X_p \in T_pM$. It is *smooth* if for every $f \in C^\infty(M)$ the map $M \to \mathbb{R},\ p \mapsto X_p(f)$ is smooth. The set of smooth vector fields on M is denoted by $\mathrm{Vect}^\infty(M)$.

Vector fields can be expressed locally: Let (U, φ) be a chart of M at p and let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. For all $i \in \{1, \dots, n\}$ define a smooth vector field $E^{(i)}$ on U by

$$E_q^{(i)} = D_{\varphi(q)}(f \circ \varphi^{-1})(e_i) \text{ for all } q \in U.$$

The smooth vector fields $(E^{(i)})_{i=1}^n$ form a basis of vector fields on U, i.e. at every $q \in U$, the tangent vectors $(E_q^{(i)})_{i=1}^n$ form a basis of T_pM. As a result, any vector field X on U can be expressed in the form

$$X_q = \sum_{i=1}^n \psi_i(q) E_q^{(i)} \text{ for all } q \in U$$

and X is smooth if and only if the functions $\psi_1, \dots, \psi_n$ are smooth. We have thus reduced the smoothness of vector fields to smoothness of functions.

With this definition of smooth vector fields, it is easy to give sense to the statement that vector fields transform smooth functions into smooth functions. To this end, recall the following definition.

Definition 2.10. Let A be a commutative algebra over a field k. A *derivation* on A is an endomorphism $\delta : A \to A$ of the underlying k-vector space A which satisfies $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.

Proposition 2.11. Let M be a smooth manifold. Then the map

$$\alpha : \mathrm{Vect}^\infty(M) \to \mathrm{End}(C^\infty(M)),\ X \mapsto (f \mapsto (Xf : p \mapsto X_pf)))$$

is an isomorphism onto its image which consists of derivations $\mathrm{Der}(C^\infty(M))$ of the $\mathbb{R}$-algebra $C^\infty(M)$.

Proof. Let $\delta : C^\infty(M) \to C^\infty(M)$ be a derivation. Then for every $p \in M$, the map $C^\infty(M) \to \mathbb{R},\ f \mapsto \delta(f)(p)$ descends to a tangent vector at p. Defining a vector field correspondingly shows that α is onto $\mathrm{Der}(C^\infty(M))$. Clearly, α is also injective. $\square$

The set $\mathrm{Der}(C^\infty(M))$ is a vector subspace of $\mathrm{End}(C^\infty(M))$ but not a subalgebra; for instance, if $M = \mathbb{R}$ and $Xf = f'$, then $X \circ X(f) = f''$ which does not satisfy Leibniz' rule. However, there is the following operation on derivations which again produces derivations and which is key in the definition of a Lie algebra.

Lemma 2.12. Let A be a commutative algebra over a field k. Let $\delta_1, \delta_2 \in \mathrm{Der}\, A$. Then $\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ is a derivation.

Proof. This is merely a computation. We have

$$\delta_1\delta_2(fg) = \delta_1\left(\delta_2(f)g + f\delta_2(g)\right) = \delta_1\delta_2(f)g + \delta_2(f)\delta_1(g) + \delta_1(f)\delta_2(g) + f\delta_1\delta_2(g)$$

and by swapping indices

$$\delta_2\delta_1(fg) = \delta_2\left(\delta_1(f)g + f\delta_1(g)\right) = \delta_2\delta_1(f)g + \delta_1(f)\delta_2(g) + \delta_2(f)\delta_1(g) + f\delta_2\delta_1(g).$$

Overall,

$$\delta_1\delta_2 - \delta_2\delta_1(fg) = (\delta_1\delta_2 - \delta_2\delta_1)(f)g + f(\delta_1\delta_2 - \delta_2\delta_1)(g).$$

□

Lemma 2.12 in particular applies to the case $A = C^\infty(M)$ and $k = \mathbb{R}$. We have thus introduced an operation which to any two vector fields associates a third one. This operation seems important, so we study it formally. The above suggests that given a commutative k-algebra A and $T_1, T_2 \in \operatorname{End}(A)$ we define $[T_1, T_2] := T_1T_2 - T_2T_1$, called the *bracket* of the two endomorphisms. Then the map

$$\operatorname{End}(A) \times \operatorname{End}(A) \to \operatorname{End}(A),\ (T_1, T_2) \mapsto [T_1, T_2]$$

is a bilinear map which preserves $\operatorname{Der}(A) \subseteq \operatorname{End}(A)$ and satisfies the following two formal properties which are readily checked

(i) (Antisymmetry). For all $T_1, T_2 \in \operatorname{End}(A)$, $[T_1, T_2] + [T_2, T_1] = 0$.
(ii) (Jacobi's identity). For all $T_1, T_2, T_3 \in \operatorname{End} A$,

$$[T_1, [T_2, T_3]] + [T_2, [T_3, T_1]] + [T_3, [T_1, T_2]] = 0$$

At this point, the identity (ii) can be seen as a substitute for associativity. To remember it, note that for the second and third bracket expression indices are permuted cyclically.

Definition 2.13. A *Lie algebra* over a field k is a k-vector space $\mathfrak{g}$ endowed with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ (x, y) \mapsto [x, y]$ which is (i) antisymmetric and satisfies (ii) Jacobi's identity.

Example 2.14. Here are some basic examples of Lie algebras.

(i) Let M be a smooth manifold. Then $\mathfrak{g} = \operatorname{Vect}^\infty(M)$ with the above bracket is a Lie algebra.
(ii) Given any associative k-algebra A, the bracket $[a, b] = ab - ba$ for all $a, b \in A$ gives a Lie algebra structure on A.
(iii) Equipping $\mathbb{R}^3$ with the cross product yields a Lie algebra. Recall that the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ associates to (u, v) the vector $u \times v$ which is orthogonal to u and v in such a way, that the tuple $(u, v, u \times v)$ is positively oriented and which has length equal to the area of the parallelogram spanned by u and v. It may be interesting to see what Jacobi's identity means geometrically in this context.

    Later on, we will associate a Lie algebra to any Lie group and $(\mathbb{R}^3, \times)$ will turn out to be isomorphic to the Lie algebra of $\mathrm{O}(3, \mathbb{R})$.

Definition 2.15. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over k. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$. A *Lie subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.

As an example, suppose M, M' are smooth manifolds and that we are given a smooth map $f : M \to M'$. Does this induce a map $f_* : \operatorname{Vect}^\infty(M) \to \operatorname{Vect}^\infty(M')$? If so, is this map a Lie algebra homomorphism? The only reasonable way to define a smooth vector field f_*X on M' given a vector field X on M and a map $f : M \to M'$ is by setting

$$(f_*X)_{m'} := D_{f^{-1}(m')}f(X_{f^{-1}(m)})$$

This however is a problem, if either f is not surjective or injective. Therefore, we restrict our attention to diffeomorphisms.

Proposition 2.16. Let M and M' be smooth manifolds and let $f : M \to M'$ be a diffeomorphism. Then the map

$$f_* : \operatorname{Vect}^\infty(M) \to \operatorname{Vect}^\infty(M'),\ X \mapsto f_*X : m' \mapsto D_{f^{-1}(m')}f(X_{f^{-1}(m')})$$

is a Lie algebra homomorphism.

For a proof, we basically have to show that f_* preserves the Lie bracket. Computing these in principal involves computing second derivatives and we urge the reader to think twice before doing that. It is often much easier to express things in terms of derivations.

Proof. (Proposition 2.16). We determine f_*X as a derivation. To this end, let $f(m) = m'$. Then for every $\varphi \in C^\infty(M')$ we have

$$(f_*X)(\varphi)(f(m)) = D_{f(m)}\varphi((f_*X)_{f(m)}) = (D_{f(m)}\varphi)(D_m f(X_m)) = D_m(\varphi \circ f)(X_m).$$

In a more condensed form, this reads

$$f_*X(\varphi) \circ f = X(\varphi \circ f) \quad \forall \varphi \in C^\infty(M').$$

Defining $f^* : C^\infty(M') \to C^\infty(M)$ by $\varphi \mapsto \varphi \circ f$, the above is

$$f^*(f_*X(\varphi)) = X(f^*\varphi) \quad \forall \varphi \in C^\infty(M')$$

which can be understood from the following commutative diagram.

$$\begin{tikzcd}
C^\infty(M') \arrow[r, "f^*"] \arrow[d, "f_*X"'] & C^\infty(M) \arrow[d, "X"] \\
C^\infty(M') \arrow[r, "f^*"] & C^\infty(M)
\end{tikzcd}$$

Since f^* is in fact an algebra isomorphism if f is a diffeomorphism, we obtain $f_*X = f^{*-1} \circ X \circ f^*$. From this, we eventually deduce that f_* is a Lie algebra homomorphism: Let $X_1, X_2 \in \operatorname{Der} C^\infty(M)$, then

$$\begin{aligned}
f_*([X_1,X_2]) &= f^{*-1}(X_1X_2 - X_2X_1)f^* = f^{*-1}X_1X_2f^* - f^{*-1}X_2X_1f^* \\
&= f^{*-1}X_1f^*f^{*-1}X_2f^* - f^{*-1}X_2f^*f^{*-1}X_1f^* \\
&= f_*X_1f_*X_2 - f_*X_2f_*X_1 = [f_*X_1, f_*X_2].
\end{aligned}$$

□

2.3. **Invariant Vector Fields and the Lie Algebra of a Lie Group.** We are now almost ready to define the Lie algebra of a Lie group. It is going to be a finite-dimensional subalgebra of the Lie algebra of all smooth vector fields which we define in the following.

Definition 2.17. Let G be a Lie group and let M be a manifold. An action of G on M is called *smooth* if the action map $G \times M \to M$ is smooth.

Retaining the above notation, observe that in particular the map $L_g : M \to M$, $m \mapsto gm = g_*m$ is a diffeomorphism with inverse $L_{g^{-1}}$. Consequently, by Proposition 2.16, the map $L_{g*} : \mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M)$ is a Lie algebra automorphism for all $g \in G$.

Definition 2.18. Let G be a Lie group, acting smoothly on a manifold M. A vector field $X \in \mathrm{Vect}^\infty(M)$ is called *G-invariant* if $L_{g*}X = X$ for all $g \in G$.

In the above situation, we denote the set of G-invariant smooth vector fields on M by $\mathrm{Vect}^\infty(M)^G$. Explicitly, X is G-invariant if for all $g \in G$ and $m \in M$ we have $D_mL_gX_m = X_{gm}$. The G-invariant smooth vector fields on M form a Lie subalgebra of the Lie algebra of all smooth vector fields.

Corollary 2.19. Let G be a Lie group, acting smoothly on a manifold M. Then the subspace $\mathrm{Vect}^\infty(M)^G \subseteq \mathrm{Vect}^\infty(M)$ is a Lie subalgebra.

Proof. We only need to show that the bracket $[-,-] : \mathrm{Vect}^\infty(M) \times \mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M)$ preserves G-invariant vector fields. This is immediate using Proposition 2.16: Let $X, Y \in \mathrm{Vect}^\infty(M)^G$. Then

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*Y}] = [X,Y]$$

for all $g \in G$ and hence $[X,Y] \in \mathrm{Vect}^\infty(M)$ is G-invariant. □

We now show that the Lie subalgebra of G-invariant smooth vector fields on a manifold M is finite-dimensional.

Lemma 2.20. Let G be a Lie group, acting smoothly and transitively on a manifold M. Then for any fixed $m_0 \in M$, the map $\mathrm{Vect}^\infty(M)^G \to T_{m_0}M,\ X \mapsto X_{m_0}$ is injective. In particular, $\mathrm{Vect}^\infty(M)^G$ is finite-dimensional.

Proof. Suppose that $X_{m_0} = 0$. Then $X_{gm_0} = D_{m_0}L_g X_{m_0} = 0$ for all $g \in G$ and hence $X \equiv 0$ since G acts transitively. □

We now apply this to the case $M = G$ and the action of G on itself by left multiplication. We denote by $\mathrm{Vect}_L^\infty(G)^G$ the space of left-invariant vector fields on G. We remark that a left-invariant vector field on a Lie group is automatically smooth.

Lemma 2.21. Let G be a Lie group. Then the map $\mathrm{Vect}_L^\infty(G)^G \to T_eG,\ X \mapsto X_e$ is an isomorphism of vector spaces. In particular, $\mathrm{Vect}_L^\infty(G)^G$ has dimension $\dim G$.

Proof. The map is injective by Lemma 2.20 and surjective by the following argument: Pick $v \in T_e(G)$, define a left-invariant (and hence smooth) vector field X on G by $X_g = D_eL_g v$. Then $X_e = v$. □

The vector field X in the proof of Lemma 2.21 is well-defined since the action of G on itself by left multiplication is free. Since further any left-invariant vector field on any Lie group is smooth, the assertion about surjectivity follows. If one in addition assumes freeness of the action in Lemma 2.20, the above isomorphism in the Lie group case carries over in the form $\mathrm{Vect}^\infty(M)^G \cong T_{m_0}M^{\mathrm{Stab}_G(m_0)}$.

Now, given any $v \in T_eG$ we denote by v^L the left-invariant vector field associated to v by the proof of Lemma 2.21.

Definition 2.22. Let G be a Lie group. The *Lie algebra* of G is the vector space T_eG endowed with the bracket

$$[v_1, v_2] := [v_1^L, v_2^L]_e \quad \forall v_1, v_2 \in T_eG$$

Note that the Lie bracket of two vectors can be computed from an arbitrarily small neighbourhood of the identity $e \in G$. Nonetheless, it involves the manifold structure of G. Does $\mathfrak{g}$ also reflect the group structure of G? As a matter of fact, it does so very well and that is why we look at this object. The group structure is hidden in the G-invariance of the vector fields.

We will now determine the Lie algebra of $G = \mathrm{GL}(n,\mathbb{R})$ and of several of its subgroups. To this end, we identify $T_{\mathrm{Id}}G$ with $M_{n,n}(\mathbb{R})$. The latter is an associative $\mathbb{R}$-algebra and hence has a Lie algebra structure by Example 2.14. This Lie algebra structure does in fact coincide with the one induced from $T_{\mathrm{Id}}\,\mathrm{GL}(n,\mathbb{R})$.

Proposition 2.23. Let $G = \mathrm{GL}(n,\mathbb{R})$. Equip $M_{n,n}(\mathbb{R})$ with the commutator Lie bracket. Then, under the identifications $M_{n,n}(\mathbb{R}) \cong T_{\mathrm{Id}}G \cong \mathrm{Vect}_L^\infty(G)^G$, the map

$$\alpha : M_{n,n}(\mathbb{R}) \to \mathrm{Vect}_L^\infty(G)^G,\ A \mapsto A^L$$

is a Lie algebra isomorphism.

Proof. We need to show that α preserves brackets. To this end, let $A, B \in M_{n,n}(\mathbb{R})$. We aim to compute $A^L B^L$ and $B^L A^L$. The tangent vector A at the identity, for instance, is given by a small arc $\gamma : (-\varepsilon, \varepsilon) \to G,\ t \mapsto \mathrm{Id} + tA$. Then for all $g \in G$, we have

$$A^L_g = \left.\frac{d}{dt}\right|_{t=0} g(\mathrm{Id} + tA) = gA$$

under the identification of $T_g \operatorname{GL}(n, \mathbb{R})$ with $M_{n,n}(\mathbb{R})$. We now compute $A^L B^L$ on $C^\infty(G)$. Let $f \in C^\infty(G)$, then for all $g \in G$ we have

$$\begin{aligned} A^L B^L(f)(g) &= \left.\frac{d}{ds}\right|_{s=0} B^L(f)(g + sgA) \\ &= \left.\frac{d^2}{ds\ dt}\right|_{(s,t)=(0,0)} f((g + sgA) + t(g + sgA)B) \\ &= \left.\frac{d^2}{ds\ dt}\right|_{(s,t)=(0,0)} f(g + \underbrace{sgA + tgB + stgAB}_{v(s,t)}). \end{aligned}$$

At this point we see again that a composition of vector fields, i.e. of derivations, is not necessarily a derivation but a second derivative. Using the Taylor expansion of f at $g \in G$ we have for small s, t:

$$f(g + v(s,t)) = f(g) + D_g f(v(s,t)) + \frac{1}{2!} D^2 f(v(s,t), v(s,t)) + \begin{array}{c}\text{multilinear forms} \\ \text{in } v(s,t)\end{array}.$$

Recall that D^2_g is a symmetric bilinear form. Hitting this expansion with $d^2/(ds\ dt)$, we obtain

$$\left.\frac{d^2}{ds\ dt}\right|_{(s,t)=(0,0)} f(g) = 0, \quad \left.\frac{d^2}{ds\ dt}\right|_{(s,t)=(0,0)} D_g f(v(s,t)) = D_g f(gAB)$$

as well as

$$\left.\frac{d^2}{ds\ dt}\right|_{(s,t)=(0,0)} D^2_g f(v(s,t), v(s,t)) = 2 D^2_g f(gA, gB)$$

using the symmetry and bilinearity of D^2_g. The higher multilinear forms in $v(s,t)$ are all going to vanish as they are sums of homogeneous polynomials in (s,t) of degree at least three. Overall, we get

$$A^L B^L(f)(g) = D_g f(gAB) + 2 D^2_g f(gA, gB)$$

and

$$B^L A^L(f)(g) = D_g f(gBA) + 2 D^2_g f(gB, gA).$$

Since D^2_g is symmetric, we obtain

$$[A^L, B^L](f)(g) = D_g(f)(g[A, B]) = [A, B]^L(f)(g).$$

This proves the proposition. □

In order to determine the Lie algebras of certain subgroups of $\operatorname{GL}(n, \mathbb{R})$, and also from a categorical view point, we would now like to know whether every homomorphism of Lie groups $\varphi : H \to G$ (e.g. an inclusion of a subgroup) gives rise to a Lie algebra homomorphism of the corresponding Lie algebras. A natural candidate would be $D_e\varphi : T_e H \to T_e G$. In the following we work towards the surprisingly subtle proof that $D_e\varphi$ is indeed a Lie algebra homomorphism.

Recall that if $f : M \to M'$ is a smooth map between smooth manifolds and if X and X' are vector fields on M and M' respectively, then X and X' are said to be

f-related, written $X \overset{f}{\sim} X'$, if $D_m f(X_m) = X'_{f(m)}$ for all $m \in M$. That is, we have the following commutative diagram:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"] & C^\infty(M) \\ C^\infty(M') \arrow[u, "f^*"] \arrow[r, "X'"] & C^\infty(M') \arrow[u, "f^*"'] \end{tikzcd}$$

Note that in general, neither of the related vector fields determines the other. There is the following standard lemma.

Lemma 2.24. Let M and M' be manifolds and let $f : M \to M'$ be a smooth map. Further, let X_1, X_2 and X'_1, X'_2 be vector fields on M and M' respectively such that $X_1 \overset{f}{\sim} X'_1$ and $X_2 \overset{f}{\sim} X'_2$. Then $[X_1, X_2] \overset{f}{\sim} [X'_1, X'_2]$.

Again it is useful to interpret things in terms of derivations in order to avoid the computation of derivates.

Proof. By definition, we have $f^*X'_1 = X_1 f^*$ and $f^*X'_2 = X_2 f^*$ as maps from $C^\infty(M')$ to $C^\infty(M)$. Consequently, we also have

$$f^*X'_1X'_2 = X_1f^*X'_2 = X_1X_2f^* \quad \text{and} \quad f^*X'_2X'_1 = X_2f^*X'_1 = X_2X_1f^*$$

Taking the difference yields

$$f^*[X'_1, X'_2] = [X_1, X_2]f^*$$

and hence $[X_1, X_2] \overset{f}{\sim} [X'_1, X'_2]$. □

Using Lemma 2.24 we now see that every smooth homomorphism of Lie groups induces a Lie algebra homomorphism.

Proposition 2.25. Let H and G be Lie groups and let $\varphi : H \to G$ be a smooth homomorphism. Then $D_e\varphi : T_eH \to T_eG$ is a Lie algebra homomorphism.

Proof. Certainly, $D_e\varphi$ is a linear map. It remains to show that it intertwines the brackets. To this end, we first show that given $v \in T_eH$, the vector fields v^L and $D_e\varphi(v)^L$ are φ-related. The assertion then follows from Lemma 2.24. Going back to the definition of two vector-fields being related, we need to show that $D_h\varphi(v^L_h) = D_e\varphi(v)^L_{\varphi(h)}$. This is merely a verification but it is important to think behind the symbols:

$$\begin{aligned} D_e\varphi(v)^L_{\varphi(h)} = D_eL_{\varphi(h)}D_e\varphi(v) = D_e(L_{\varphi(h)} \circ \varphi)(v) = \\ = D_e(\varphi \circ L_h)(v) = D_h\varphi D_eL_h(v) = D_h\varphi(v^L_h). \end{aligned}$$

Now, given $v_1, v_2 \in T_eH$ we know that $v^L_1 \overset{\varphi}{\sim} D_e\varphi(v_1)^L$ and $v^L_2 \overset{\varphi}{\sim} D_e\varphi(v_2)^L$ and hence $[v^L_1, v^L_2] \overset{\varphi}{\sim} [D_e\varphi(v_1)^L, D_e\varphi(v_2)^L]$ by Lemma 2.24. In particular,

$$D_e\varphi([v_1, v_2]) = [D_e\varphi(v_1), D_e\varphi(v_2)].$$

□

Corollary 2.26. Let G be a Lie group and let $H \leq G$ be a subgroup of G which is also a regular submanifold. Then the inclusion $H \rightarrowtail G$ realizes T_eH as a Lie subalgebra of T_eG.

Example 2.27. Together with Proposition 2.23, Corollary 2.26 enables us to determine the Lie algebras of various Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

(i) $\mathrm{SL}(n,\mathbb{R})$. The Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is

$$\mathfrak{sl}(n,\mathbb{R}) := \{X \in M_{n,n}(\mathbb{R}) \mid \operatorname{tr} X = 0\}.$$

To see this, recall that we have realized $\mathrm{SL}(n,\mathbb{R})$ as $\det^{-1}(1) \leq \mathrm{GL}(n,\mathbb{R})$ in Example 2.7. Then

$$T_{\mathrm{Id}}\,\mathrm{SL}(n,\mathbb{R}) = \ker D_{\mathrm{Id}} \det = \ker \operatorname{tr} = \{X \in M_{n,n}(\mathbb{R}) \mid \operatorname{tr} X = 0\}.$$

(ii) $\mathrm{O}(n,\mathbb{R})$. The Lie algebra of $\mathrm{O}(n,\mathbb{R})$ is

$$\mathfrak{o}(n,\mathbb{R}) := \{X \in M_{n,n}(\mathbb{R}) \mid X + X^T = 0\}.$$

Again, recall that we have realized $\mathrm{O}(n,\mathbb{R})$ as $f^{-1}(\mathrm{Id})$ in Example 2.7, where $f : \mathrm{GL}(n,\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ A \mapsto A^T A$. Then

$$\begin{aligned} T_{\mathrm{Id}}\mathrm{O}(n,\mathbb{R}) &= \ker D_{\mathrm{Id}} f = \ker(X \mapsto X + X^T) \\ &= \{X \in M_{n,n}(\mathbb{R}) \mid X + X^T = 0\}. \end{aligned}$$

(One can check by hand that the commutator of antisymmetric matrices is indeed antisymmetric — the theory works!)

(iii) Unipotent upper-triangular matrices. The Lie algebra of

$$N = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \forall\, 1 \leq i \leq n :\ x_{ii} = 1, \\ \forall\, 1 \leq j < i \leq n :\ x_{ij} = 0 \end{array} \right\}$$

is given by

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & 0 \end{pmatrix} \;\middle|\; \forall\, 1 \leq j \leq i \leq n :\ x_{ij} = 0 \right\}$$

as can be checked directly by determining tangent vectors in the classical way using curves through $\mathrm{Id} \in N$.

(vi) Diagonal matrices. The Lie algebra of

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \;\middle|\; \forall\, 1 \leq i \leq n :\ \lambda_i \in \mathbb{R} - \{0\} \right\}$$

is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \;\middle|\; \forall\, 1 \leq i \leq n :\ \lambda_i \in \mathbb{R} \right\}.$$

Again, this can be checked directly.

Example 2.28. $\mathrm{GL}(n,\mathbb{C})$. In order to determine the Lie algebra of $\mathrm{GL}(n,\mathbb{C})$ we cannot simply realize $\mathrm{GL}(n,\mathbb{C})$ as a subset of $\mathrm{GL}(2n,\mathbb{R})$ because as such it is not open and hence does not have Lie algebra $\mathfrak{gl}(2n,\mathbb{R})$. However, if we consider $M_{n,n}(\mathbb{C})$ as a $2n^2$-dimensional real vector space, then $\mathrm{GL}(n,\mathbb{C})$ is an open subset of this and hence inherits the structure of a smooth (real) manifold. Further, the proof of Proposition 2.23 carries over to this setting. Then $\mathrm{SL}(n,\mathbb{C}) = \det^{-1}(1)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$. Note however, that there are subgroups of $\mathrm{GL}(n,\mathbb{C})$ which are not defined by polynomial equations, e.g.

$$\mathrm{U}(n,\mathbb{C}) := \{A \in \mathrm{GL}(n,\mathbb{C}) \mid \overline{A}^T A = \mathrm{Id}\}.$$

It is natural to consider the map $f : \mathrm{GL}(n,\mathbb{C}) \to M_{n,n}(\mathbb{C}),\ A \mapsto \overline{A}^T A$ which is a smooth map between real vector spaces and as such polynomial but it is not polynomial over $\mathbb{C}$ as it involves the non-polynomial conjugation $\mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$. Using $D_{\mathrm{Id}} f = X + \overline{X}^T$ one shows that f has constant real rank n^2 on $\mathrm{GL}(n,\mathbb{C})$. Hence $\mathrm{U}(n,\mathbb{C})$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$.

Returning to the categorical view point, we have established the *Lie functor* from the category of Lie groups and smooth homomorphisms **LieGrp** to the category of real, finite-dimensional Lie algebras and Lie algebra homomorphisms **LieAlg**:

$$\mathrm{Lie} : \mathbf{LieGrp} \to \mathbf{LieAlg}.$$

The fundamental question of course is how much information we loose when passing from to Lie group to its Lie algebra which after all is a finite-dimensional vector space and hence potentially much easier to understand than the Lie group.

(i) Is it possible to go back in the sense that every finite-dimensional Lie algebra comes from a Lie group? The answer here is yes, but it is quite subtle.

(ii) (Faithfulness). Is a Lie group uniquely determined by its Lie algebra? The answer is no: If G is a Lie group and F is any finite group, then $G' := G \times F$, which has as many connected components as F has elements, has the same tangent space at the identity as G whose Lie algebra structure is determined by an arbitrarily small neighbourhood of the identity.

But even a connected Lie group is still not determined by its Lie algebra. Consider for instance the case of $\mathbb{R}$ and S^1 or, for a two-dimensional example, $\mathbb{R}^2$ and $\mathbb{R}^2 / \mathbb{Z}^2$. Here, the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2 / \mathbb{Z}^2$ is a covering map and it is readily checked that it induces a correspondence between invariant vector-fields on the two Lie groups. This is the general phenomenon in the sense of the following statement: Let G_1 and G_2 be connected Lie groups with isomorphic Lie algebras. Then the universal covers $\tilde{G}_1$ and $\tilde{G}_2$ of G_1 and G_2 respectively are isomorphic. In particular, a connected simply connected Lie group is uniquely determined by its Lie algebra.

(iii) It is desirable that the category of Lie groups is closed under certain operations on groups. For instance, we have seen that a product of Lie groups is again a Lie group in a natural fashion. What about, say, the center $Z(G)$ of a Lie group G. A priori it is just a closed subgroup of G. Is it also a regular submanifold and hence a Lie subgroup? We will prove the following striking theorem of Cartan: If H is a closed subgroup of a Lie group G, then H is a regular submanifold and hence a Lie subgroup.

2.4. **The Exponential Map.** The exponential map is a basic tool which links a Lie group and its Lie algebra. We will treat the (important) case of $G := \mathrm{GL}(n,\mathbb{R})$ first and thereafter move to the general case. Let $A \in M_{n,n}(\mathbb{R}) \cong T_{\mathrm{Id}}\,\mathrm{GL}(n,\mathbb{R})$ be a tangent vector to G at the identity and denote by A^L the associated left-invariant vector field, namely $A^L_g = gA$ for all $g \in G$. Then it is natural to study the integral curves of this vector field. Recall that by definition, an integral curve for A^L through $g \in G$ is a smooth map $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = g$ and $\dot\gamma(t) = A^L_{\gamma(t)}$. This is a first order system of ODE's with constant coefficients. If $n = 1$, clearly $\gamma(t) = ge^{At}$ is a solution. This suggests defining an exponential of matrices to deal with the higher-dimensional case:

$$\mathrm{Exp} : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ A \mapsto \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

There is, of course, the issue of convergence of the above series. To resolve it, choose a submultiplicative norm on $M_{n,n}(\mathbb{R})$, e.g. the operator norm $\| - \|_{\mathrm{op}}$ associated to a norm $\| - \|$ on $\mathbb{R}^n$ defined by

$$\|A\|_{\mathrm{op}} := \sup_{\|v\| \leq 1} \|Av\| \quad (A \in M_{n,n}(\mathbb{R})).$$

This norm is submultiplicative in the sense that $\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}}$ for all $A, B \in M_{n,n}(\mathbb{R})$. As there will be no reason to confuse $\| - \|_{\mathrm{op}}$ with $\| - \|$ we drop the subscript "op" in the following.

Lemma 2.29.

(i) Let $A \in M_{n,n}(\mathbb{R})$. The partial sums $\sum_{n=0}^{N} A^n/n!,\ N \in \mathbb{N}$ converge uniformly on balls of finite radius to a smooth map, called Exp.

(ii) For all $A, B \in M_{n,n}(\mathbb{R})$ with $[A, B] = 0$ the exponential satisfies

$$\mathrm{Exp}(A + B) = \mathrm{Exp}(A)\,\mathrm{Exp}(B)$$

(iii) For all $A \in M_{n,n}(\mathbb{R})$, the map $\varphi_A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R}),\ t \mapsto \mathrm{Exp}(tA)$ is a smooth homomorphism with $\dot{\varphi}(0) = A$.

(iv) Any smooth homomorphism $\psi : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ is of the form $\psi(t) \equiv \varphi_A(t)$ where $A := \dot{\psi}(0)$.

Proof. To prove uniform convergence on balls of finite radius, note that

$$\left\| \mathrm{Exp}\, A - \sum_{n=0}^{N} \frac{A^n}{n!} \right\| = \left\| \sum_{n=N+1}^{\infty} \frac{A^n}{n!} \right\| \leq \sum_{n=N+1}^{\infty} \frac{\|A\|^n}{n!} \leq \sum_{n=N+1}^{\infty} \frac{R^n}{n!}$$

for all $A \in M_{n,n}(\mathbb{R})$ with $\|A\| \leq R$. In order to show the continuity of partial derivatives of Exp, we need to understand the partial derivatives of the map $M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ X \mapsto X^n$. For instance,

$$\frac{\partial}{\partial x_{ij}} X^n = \sum_{k_1 + k_2 = n-1} X^{k_1} E_{ij} X^{k_2}$$

by applying the product rule. Therefore

$$\left\| \frac{\partial}{\partial x_{ij}} X^n \right\| \leq n \|X\|^{n-1}$$

which establishes continuity of the first partial derivatives of Exp. Iterate this to deal with higher order derivatives.

For part (ii), note that if $[A, B] = 0$ then

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

by the binomial theorem and hence

$$\begin{aligned}\operatorname{Exp}(A+B) &= \sum_{n=0}^{\infty}\frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1+k_2=n}\binom{n}{k_1}A^{k_1}B^{k_2} = \\ &= \sum_{n=0}^{\infty}\sum_{k_1+k_2=n}\frac{1}{k_1!k_2!}A^{k_1}B^{k_2} = \sum_{k_1,k_2=0}^{\infty}\frac{A^{k_1}}{k_1!}\frac{B^{k_2}}{k_2!} = \\ &= \left(\sum_{k_1=0}^{\infty}\frac{A^{k_1}}{k_1!}\right)\left(\sum_{k_2=0}^{\infty}\frac{B^{k_2}}{k_2!}\right)\end{aligned}$$

As to (iii), we compute for all $s,t\in\mathbb{R}$ using (ii):

$$\varphi_A(s+t) = \operatorname{Exp}(tA+sA) = \operatorname{Exp}(tA)\operatorname{Exp}(sA) = \varphi_A(t)\varphi_A(s).$$

Further, expanding $\varphi_A(t) = \operatorname{Id}+tA+t^2A^2/2!+\cdots$, we see that $d/dt|_{t=0}\varphi_A(t) = A$.

For part (iv), we note that

$$\dot\psi(t) = \frac{d}{ds}\bigg|_{s=0}\psi(t+s) = \frac{d}{ds}\bigg|_{s=0}\psi(t)\psi(s) = \psi(t)A.$$

Therefore, ψ is an integral curve of A^L through Id_n. By the uniqueness of integral curves, $\psi(t)=\varphi_A(t)$ for all $t\in\mathbb{R}$. □

Example 2.30. Here are examples of exponentials of matrices.

(i) The exponential of a diagonal matrix $D = \operatorname{diag}(\lambda_1,\dots,\lambda_n)\in M_{n,n}(\mathbb{C})$ does indeed look like an exponential:

$$\operatorname{Exp} tD = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n}\end{pmatrix}.$$

(ii) The exponential of a nilpotent matrix looks more like a polynomial. For instance if $N = E_{12}+E_{23}\in M_{3,3}(\mathbb{C})$, then

$$\begin{aligned}\operatorname{Exp} tN &= \operatorname{Id}+N+N^2/2!+N^3/3!+\dots \\ &= \operatorname{Id}+\begin{pmatrix}0 & t & \\ & 0 & t\\ & & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & t^2/2\\ & 0 & 0\\ & & 0\end{pmatrix} = \begin{pmatrix}1 & t & t^2/2\\ & 1 & t\\ & & 1\end{pmatrix}.\end{aligned}$$

(iii) In general, a matrix $A\in M_{n,n}(\mathbb{C})$ admits a Jordan decomposition $A = TJT^{-1}$ for some $T\in\operatorname{GL}(n,\mathbb{C})$ and $J = D+N$ for a diagonal matrix $D\in M_{n,n}(\mathbb{C})$ and a nilpotent matrix $N\in M_{n,n}(\mathbb{C})$ such that D and N commute. Then for all $t\in\mathbb{R}$,

$$\begin{aligned}\operatorname{Exp} tA &= \operatorname{Exp} t(TJT^{-1}) = T\operatorname{Exp}(tJ)T^{-1} = \\ &= T\operatorname{Exp}(tD+tN)T^{-1} = T\operatorname{Exp}(tD)\operatorname{Exp}(tN)T^{-1}\end{aligned}$$

by Lemma 2.29.

We now recall some facts from the general theory of smooth vector fields and their integral curves in order to define an exponential map for every Lie group. As a general reference, we recommend [Boo86].

Definition 2.31. Let M be a manifold and let X be a vector field on M. An integral curve of X through $m\in M$ is a smooth map $\gamma:(-\delta,\delta)\to M$ such that $\dot\gamma(t) = X_{\gamma(t)}$ and $\gamma(0)=m$.

Here is the existence and uniqueness statement regarding integral curves.

Theorem 2.32. Let M be a manifold and let X be a vector field on M.

(i) (Existence). For all $p \in M$, there is $\delta > 0$ and an open neighbourhood V of p such that for all $q \in V$, there is an integral curve $\gamma_q : I_\delta \to M$ of X through q.
(ii) (Uniqueness). Any two integral curves of X through q coincide on their domain of definition.
(iii) (Smooth dependance on initial condition). The map $I_\delta \times V \to M,\ (t, q) \mapsto \gamma_q(t)$ is smooth.

We are only going to deal with integral curves that are defined for all times. In particular, left-invariant vector fields on Lie groups have this property.

Definition 2.33. Let M be a manifold and let X be a vector field on M. Then X is *complete* if for all $q \in M$ there is an integral curve of X through q defined on $\mathbb{R}$.

For a complete vector field X on M, we obtain a smooth one-parameter group of diffeomorphisms $\Phi^X : \mathbb{R} \times M \to M,\ (t, m) \mapsto \gamma_m(t)$, termed the *flow* of X, with the property $\Phi(t_1 + t_2, m) = \Phi(t_1, \Phi(t_2, m))$ for all $t_1, t_2 \in \mathbb{R}$ and $m \in M$. Indeed, the map $t \mapsto \gamma_m(t_2 + t)$ is an integral curve of X going through $\gamma_m(t_2)$. By uniqueness, $\gamma_m(t_2 + t) = \gamma_{\gamma_m(t_2)}(t)$. Reformulating this in terms of Φ yields the assertion.

Using the flow of vector fields we can give a geometric interpretation of what it means that the bracket of two vector fields vanishes.

Lemma 2.34. Let M be a manifold. Further, let X and Y be smooth complete vector fields on M and denote by $\Phi^X : \mathbb{R} \times M \to M$ and $\Phi^Y : \mathbb{R} \times M \to M$ the corresponding flows. Then the following statements are equivalent:

(i) $[X, Y] = 0$.
(ii) For all $s, t \in \mathbb{R},\ m \in M$: $\Phi^X_t \circ \Phi^Y_s(m) = \Phi^Y_s \circ \Phi^X_t(m)$

Here, we use the notation $\Phi^X_t(m) := \Phi^X(t, m)$ and $\Phi^Y_s(m) := \Phi^Y(s, m)$.

A proof of Lemma 2.34 can be found in [Boo86]. Turning to Lie groups now, we have the following.

Theorem 2.35. Let G be a Lie group. Then the following hold.

(i) Left-invariant vector fields on G are complete.
(ii) For every $v \in T_eG$, let v^L be the associated left-invariant vector field and let $\varphi_v : \mathbb{R} \to G$ be the integral curve of v^L through $e \in G$. Then φ_v is a smooth homomorphism, i.e. $\varphi_v(t_1 + t_2) = \varphi_v(t_1)\varphi_v(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.
(iii) The flow $\Phi : \mathbb{R} \times G \to G$ of v^L is given by $\Phi(t, g) = g\varphi_v(t)$.

Note, that in Theorem 2.35, part (ii) and (iii) already assume part (i). The proof is not very difficult and illustrates that left-invariance is key.

Proof. For part (i), we note the following: Let $v \in \mathfrak{g}$. If $\gamma : I_\delta \to G$ is an integral curve of v^L through e and $\gamma_g : I_\delta \to G$ is defined by $\gamma_g(t) = g\gamma(t)$, then γ_g is an integral curve of v^L through g. Clearly, $\gamma_g(0) = g$ and

$$\dot\gamma_g(t) = D_{\gamma(t)}L_g(\dot\gamma(t)) = D_{\gamma(t)}L_g(v^L_{\gamma(t)}) = v^L_{g\gamma(t)} = v^L_{\gamma_g(t)}.$$

Now, let $I \subseteq \mathbb{R}$ be the largest interval where $\gamma : I \to G$ is defined. For every $t_0 \in I$, the curve $\gamma_{\gamma(t_0)}$ is an integral curve of v^L through $\gamma(t_0)$ defined on the interval I_δ. By uniqueness, $(t_0 - \delta, t_0 + \delta) \subseteq I$ and hence $I = \mathbb{R}$, i.e. v^L is complete.

The above reasoning also implies that $\Phi(t, g) = g\Phi(t, e)$ for all $t \in \mathbb{R}$ and $g \in G$. This is (iii). Combining it with a general property of flows, we obtain

$$\Phi(t_1 + t_2, g) = \Phi(t_1, \Phi(t_2, g)) = \Phi(t_2, g)\Phi(t_1, e)$$

for all $t_1, t_2 \in \mathbb{R}$ and $g \in G$ and hence

$$\Phi(t_1 + t_2, e) = \Phi(t_2 + t_1) = \Phi(t_1, e)\Phi(t_2, e) \quad \Leftrightarrow \quad \varphi_v(t_1 + t_2) = \varphi_v(t_2)\varphi_v(t_1)$$

by evaluating at $g = e$ which is (ii). ☐

From the above, we see that elements of the Lie algebra of a Lie group give rise to smooth homomorphisms from $\mathbb{R} \to G$.

Definition 2.36. Let G be a Lie group. A smooth homomorphism from $\mathbb{R}$ to G is a *one-parameter group.*

Corollary 2.37. Let G be a Lie group. If $\varphi : \mathbb{R} \to G$ is a one-parameter group, then $\varphi \equiv \varphi_v$ for $v = \dot{\varphi}(0) \in \mathfrak{g}$ and $\varphi_{sv}(t) = \varphi_v(st)$ for all $s \in \mathbb{R}$.

Proof. Let $v = \dot{\varphi}(0)$ and let v^L be the associated left-invariant vector field. Since $\varphi(t+s) = \varphi(s)\varphi(t)$ for all $s, t \in \mathbb{R}$, we have

$$\dot{\varphi}(s) = \frac{d}{dt}\bigg|_{t=0} \varphi(t+s) = D_e L_{\varphi(s)}(\dot{\varphi}(0)) = D_e L_{\varphi(s)}(v) = v^L_{\varphi(s)}.$$

This implies that φ is an integral curve of v^L through e and hence $\varphi \equiv \varphi_v$.

For example, we may consider the one-parameter subgroup $\mathbb{R} \to G,\ t \mapsto \varphi_v(st)$ for fixed $s \in \mathbb{R}$. Its tangent vector at $t = 0$ is sv. Hence $\varphi_v(st) = \varphi_{sv}(t)$. □

You may wonder what the purpose of introducing a definition is when immediately after one learns that one hasn't introduced anything new. In this case, one may for instance look at one-parameter subgroups in a topological rather than smooth setting, or vary the group of parameters. Related to this is that one-parameter subgroups may be viewed as an integrated form of a vector field that one can work with without any explicit reference to derivatives. Also, one could develop the whole theory of manifolds and Lie groups over the field $\mathbb{Q}_p$ rather than $\mathbb{R}$ and then use one-parameter subgroups to define a Lie algebra.

Anyway, we are now able to define the exponential map in general.

Definition 2.38. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *exponential map* $\exp_G : \mathfrak{g} \to G$ is defined by $v \mapsto \varphi_v(1)$ where φ_v is the integral curve of v^L through $e \in G$.

Corollary 2.39. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G : \mathfrak{g} \to G$. Then:

(i) $\exp_G(tv) = \varphi_v(t)$ for all $v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

(ii) Let $v, w \in \mathfrak{g}$. Then $\exp_G(v + w) = \exp_G(v) \exp_G(w)$ if $[v, w] = 0$.

Proof. For (i), note that $\exp(tv) = \varphi_{tv}(1) = \varphi_v(t)$ by Corollary 2.37. Part (ii) is an immediate consequence of Lemma 2.34. □

The following corollary provides an explicit link between a smooth homomorphism of Lie groups and its derivative which we express in a commutative diagram. These are sometimes more telling than the associated formulae.

Corollary 2.40. Let G, H be Lie groups and let $\psi : G \to H$ be a smooth homomorphism. Then the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\psi"] & H \\ \mathfrak{g} \arrow[u, "\exp_G"] \arrow[r, "D_e\psi"'] & \mathfrak{h} \arrow[u, "\exp_H"'] \end{tikzcd}$$

The proof is exceedingly simple.

Proof. For $v \in \mathfrak{g}$, the map $\gamma : \mathbb{R} \to H,\ t \mapsto \psi(\exp_G(tv))$ is a one-parameter subgroup of H. Its derivative at $0 \in \mathbb{R}$ is given by $\dot{\gamma}(0) = D_e\psi(v) \in \mathfrak{h}$. We therefore have $\gamma(t) = \exp_H(tD_e\psi(v))$ by Corollary 2.37 and Definition 2.38 which implies $\psi(\exp_G(tv)) = \exp_H(tD_e\psi(v))$. Now evaluate at $t = 1$. □

Lemma 2.40 hints at the potential of “integrating“ a homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ to one of the corresponding groups G and H. In this direction, we now try to understand the image of the exponential map. In general, this is difficult to determine, but there is always the following.

Theorem 2.41. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The differential $D_0 \exp_G$ of $\exp_G : \mathfrak{g} \to G$ at $0 \in \mathfrak{g}$ is the identity on $\mathfrak{g}$. As a consequence, there is an open neighbourhood U of $0 \in \mathfrak{g}$ such that $\exp_G(U)$ is open in G and the restriction $\exp_G|_U : U \to \exp_G(U)$ is a diffeomorphism.

In this case, the above result is called a theorem because it is important rather than because its proof is complicated, which in fact is anything but, assuming the inverse function theorem.

Proof. The second assertion follows from the first and the inverse function theorem. For the first assertion, let $\xi \in \mathfrak{g}$ be a tangent vector to $\mathfrak{g}$ at $0 \in \mathfrak{g}$. Then

$$D_0 \exp_G(\xi) = \left.\frac{d}{dt}\right|_{t=0} \exp_G(t\xi) = \xi.$$

□

In particular, Theorem 2.41 provides us with a canonical chart of the Lie group at the identity. Transfering it around by left translation, provides a chart at every point. What more can be said about (the image of) the exponential map? Is it always surjective?

Example 2.42. Here are several examples of Lie groups for which the exponential map is surjective, but also examples for which it is not.

(i) The following is due to E. Cartan: If K is a connected, compact Lie group, then $\exp_K$ is surjective. A proof could work as follows: Every compact, connected Lie group K admits a bi-invariant Riemannian metric. The bi-invariance implies that the Lie group exponential coincides with the Riemannian exponential and then surjectivity follows from the Hopf-Rinow Theorem.

Bi-invariance is crucial here. An arbitrary Lie group admits both a left- and a right-invariant Riemannian metric but not necessarily a bi-invariant one in which case the Lie group exponential does not necessarily coincide with a Riemannian exponential.

As an example, the exponential map $\mathfrak{u}(n, \mathbb{C}) \to \mathrm{U}(n, \mathbb{C})$ is surjective. This can be seen in a more elementary fashion than the above one. Namely, it follows from the series definition of the exponential for matrix groups that $g \operatorname{Exp} A g^{-1} = \operatorname{Exp}(gAg^{-1})$ for all $A \in \mathfrak{u}(n, \mathbb{C})$ and $g \in U(n, \mathbb{C})$, i.e. the image of Exp is closed under conjugation. Also, every unitary matrix $A \in \mathrm{U}(n, \mathbb{C})$ is diagonalizable as

$$A = \begin{pmatrix} e^{i\varphi_1} & & \\ & \vdots & \\ & & e^{i\varphi_n} \end{pmatrix} = \operatorname{Exp} \begin{pmatrix} i\varphi_1 & & \\ & \vdots & \\ & & i\varphi_n \end{pmatrix}$$

for some $\varphi_1, \dots, \varphi_n \in \mathbb{R}$. Combined with conjugation, this implies that for $G = \mathrm{U}(n, \mathbb{C})$, the exponential map is surjective. This can actually be made into a proof of the general case if one knows some structure theory of Lie groups, in particular maximal tori.

(ii) The exponential map $\operatorname{Exp} : \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is surjective. To see this, one can use a similar argument as above. Namely, in addition to conjugation invariance of the image of Exp, every matrix $A \in \mathrm{GL}(n, \mathbb{C})$ is equivalent to a Jordan matrix $J = D + N$ where D is a diagonal matrix and N is a unipotent upper-triangular matrix such that D and N commute. The matrix D is in the image of the exponential map as above. The following example deals with unipotent upper-triangular matrices.

(iii) For the Lie group

$$N = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & 1 \end{pmatrix} \middle| \begin{array}{l} \forall\, 1 \le i \le n: \ x_{ii} = 1, \\ \forall\, 1 \le j < i \le n: \ x_{ij} = 0 \end{array} \right\}$$

with Lie algebra

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & 0 \end{pmatrix} \middle| \forall\, 1 \le j \le i \le n: \ x_{ij} = 0 \right\}$$

the exponential map is surjective: Note that $X \in \mathfrak{n}$ satisfies $X^n = 0$ and hence

$$\operatorname{Exp} X = \operatorname{Id} + X + \frac{X^2}{2} + \cdots + \frac{X^{n-1}}{(n-1)!}$$

is a polynomial expression. One could prove surjectivity by finding a right-inverse for Exp. A candidate certainly is the "natural logarithm". The power series expansion of $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ at $x = 1$ is given by

$$\ln(x) = \sum_{k=1}^{\infty} (-1)^{n+1} \frac{(x-1)^k}{k}.$$

Now, if $g \in N$, then $g - \operatorname{Id}$ is nilpotent of degree at most n, i.e. $(g - \operatorname{Id})^n = 0$. Therefore

$$\operatorname{Log} : N \to \mathfrak{n},\ g \mapsto (g - \operatorname{Id}) - \frac{(g-\operatorname{Id})^2}{2} + \frac{(g-\operatorname{Id})^3}{3} - \cdots + (-1)^n \frac{(g-\operatorname{Id})^{n-1}}{n-1}$$

is an explicit inverse for Exp. Hence $\operatorname{Exp} : \mathfrak{n} \to N$ is surjective.

(iv) For $\mathrm{SL}(2,\mathbb{R})$, the exponential map is not surjective. To show this, one might suspect, that $-\operatorname{Id}$ is not contained in the image of $\operatorname{Exp} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ because -1 is not contained in the image of the real exponential. But $-\operatorname{Id}$ is perfectly contained in the one-parameter group

$$\mathbb{R} \to \mathrm{SO}(2,\mathbb{R}) \le \mathrm{SL}(2,\mathbb{R}),\ \theta \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and hence in the image of the exponential map. In fact, if $g \in \mathrm{SL}(2,\mathbb{R})$ is not contained in the image of the exponential map, it must not be contained in a compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ by part (i). Here, however, $-\operatorname{Id} \in \mathrm{SO}(2,\mathbb{R}) \le \mathrm{SL}(2,\mathbb{R})$.

Anyway, the matrix

$$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

is not contained in the image of the exponential map which can be seen as follows: Any matrix $A \in \operatorname{Exp}(\mathfrak{sl}(2,\mathbb{R}))$ can be written as a square of another matrix $B \in \mathrm{SL}(2,\mathbb{R})$. In fact, if $A = \operatorname{Exp} X$, we note that $\operatorname{Exp}(X/2 + X/2) = \operatorname{Exp}(X/2)^2$ where $X/2 \in \mathfrak{sl}(2,\mathbb{R})$. However, the above matrix is not a square as can be checked directly. Also, the following holds: Any $B \in \mathrm{SL}(2,\mathbb{R})$ satisfies its characteristic polynomial $\chi_B(t) = t^2 - (\operatorname{tr} B)t + \det B$, i.e.

$$0 = B^2 - (\operatorname{tr} B)B + (\det B)\,\mathrm{Id}$$

and hence by taking the trace:

$$0 = \operatorname{tr}(B^2) - (\operatorname{tr} B)^2 + 2$$

If $A = B^2$, this implies $\operatorname{tr}(A) \geq -2$ and hence we conclude for instance that

$$\begin{pmatrix} -2 & \\ & -1/2 \end{pmatrix} \notin \operatorname{Exp}(\mathfrak{sl}(2,\mathbb{R})).$$

A further consequence of the above is the following.

Corollary 2.43. Let G be a connected abelian Lie group with Lie algebra $\mathfrak{g}$. Then $\exp_G : \mathfrak{g} \to G$ is a smooth surjective homomorphism. The kernel $\Gamma := \ker \exp_G \leq \mathfrak{g}$ is discrete and $\exp_G$ induces an isomorphism of Lie groups $\exp_G : \mathfrak{g}/\Gamma \cong G$.

For the proof, we are going to need the following remark, the details of which to work out is left as an exercise.

Remark 2.44. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $T_{(e,e)}(G\times H) = \mathfrak{g} \times \mathfrak{h}$ by differential geometry. The Lie algebra structure on $T_{(e,e)}(G \times H) = \mathfrak{g} \times \mathfrak{h}$ is given by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$

for all $X_1, X_2 \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{h}$. Simply look at vector fields on product manifolds and how their brackets are expressed in terms of their components.

Finally, $\exp_{G\times H} : \mathfrak{g} \times \mathfrak{h} \to G \times H$ is given by $(X, Y) \mapsto (\exp_G X, \exp_H Y)$ for all $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$. In fact, any one-parameter group $\varphi : \mathbb{R} \to G \times H$ is of the form $\varphi(t) = (\psi(t), \eta(t))$ where $\psi : \mathbb{R} \to G$ and $\eta : \mathbb{R} \to H$ are one-parameter groups in G and H respectively.

Proof. (Corollary 2.43). First, we make the following observation: The product $m : G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ is a smooth *homomorphism* if (and only if) G is abelian.

Now, by Corollary 2.40, we have the following commutative diagram:

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G \\ \mathfrak{g} \times \mathfrak{g} \arrow[u, "\exp_{G\times G}"] \arrow[r, "D_{(e,e)}m"'] & \mathfrak{g} \arrow[u, "\exp_G"'] \end{tikzcd}$$

Hence for all $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} \exp_G X \exp_G Y &= m((\exp_G X, \exp_G Y)) \\ &= m(\exp_{G\times G}(X, Y)) = \exp_G D_{(e,e)}m(X, Y). \end{aligned}$$

We now determine the right-hand expression. Observe that $D_{(e,e)}m : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a linear map, hence $D_{(e,e)}m(X, Y) = D_{(e,e)}m(X, 0) + D_{(e,e)}m(0, Y)$. We further note that

$$D_{(e,e)}m(X, 0) = D_{(e,e)}m \circ D_e i_1(X)$$

where $i_1 : G \to G \times G$, $g \mapsto (g, e)$ is the canonical injection associated to the first coordinate. Now $(m \circ i_1)(g) = m(g, e) = ge = g$ for all $g \in G$ and hence $m \circ i_1 = \mathrm{id}_G$ whence $D_e(m \circ i_1)(X) = X$ for all $X \in \mathfrak{g}$. Overall, we deduce that $D_{(e,e)}m(X, Y) = X + Y$ and therefore

$$\exp_G X \exp_G Y = \exp_G(X + Y)$$

for all $X, Y \in \mathfrak{g}$, i.e. $\exp_G$ is a homomorphism.

As to surjectivity of $\exp_G$, we already know that $\operatorname{im}\exp_G$ is a neighbourhood of $e \in G$. Since $\exp_G$ is a homomorphism by the above, $\operatorname{im}\exp_G$ is also an open subgroup, hence closed and therefore equal to the whole of G since G is connected.

Eventually, let $\Gamma := \ker\exp_G$. We know that there is an open neighbourhood of $0 \in \mathfrak{g}$ such that $\exp_G|_U$ is a diffeomorphism onto its image. In particular $U \cap \Gamma = 0$ by injectivity, hence Γ is discrete. Then $\pi : \mathfrak{g} \to \mathfrak{g}/\Gamma$ is a covering map via which one equips $\mathfrak{g}/\Gamma$ with a Lie group structure and then uses the implicit function theorem to deduce that $\exp_G$ descends to a Lie group isomorphism $\mathfrak{g}/\Gamma \cong G$. □

Any discrete subgroup of $\mathbb{R}^n$ is in fact of the form $\Gamma = \mathbb{Z}\, v_1 + \cdots + \mathbb{Z}\, v_r$ for some linearly independent vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ and then

$$\mathbb{R}^n/\Gamma \cong \mathbb{R}^n/(\mathbb{Z}^r \times 0) \cong (\mathbb{R}^n/\mathbb{Z}^r) \times \mathbb{R}^{n-r} \cong (S^1)^r \times \mathbb{R}^{n-r}.$$

Hence these are the only connected abelian Lie groups.

Another interesting consequence of the properties of the exponential map is the following.

Corollary 2.45. Let G be a Lie group. Then there is an open neighbourhood of $e \in G$ which does not contain any non-trivial subgroup of G.

As a matter of fact, the above "no small subgroups" property characterizes Lie groups among locally compact topological groups.

Proof. Pick a norm $\|-\|$ on $\mathfrak{g}$ and $r > 0$ such that

$$\exp_G|_{B(0,r)} : B(0,r) \to \exp_G(B(0,r))$$

is bijective (or even a diffeomorphism). Suppose that $L \leq G$ is a subgroup of G which is contained in $\exp_G(B(0,r/2))$. If $L \neq \{e\}$ we pick $u \in B(0,r/2) - \{0\}$ such that $\exp_G(u) \in L$. Since L is a subgroup, this implies for all $k \in \mathbb{Z}$ that $\exp_G(ku) = (\exp_G u)^k \in L$. On the other hand,

$$0 \leq \|u\| < r/2 \quad \Rightarrow \quad \exists n \in \mathbb{N} : \ r/2 < n\|u\| < r$$

and hence $nu \in B(0,r) - B(0,r/2)$. But then also $\exp_G(nu) \in \exp_G(B(0,r)) - \exp_G(B(0,r/2))$ which is a contradiction to $L \subseteq \exp_G(B(0,r/2))$. □

As a side remark, note that our proof of Corollary 2.45 critically uses the archimedean property

$$0 < \|u\| < r/2 \quad \Rightarrow \quad \exists n \in \mathbb{N} : \ r/2 < n\|u\| < r$$

of $\|-\|$ which is not valid for, e.g., the p-adic norm on $\mathbb{Q}_p$; and in fact, p-adic Lie groups may have small subgroups.

2.5. **Cartan's Theorem on Closed Subgroups.** We now prove the following theorem of Cartan the significance of which we have already pointed out in the discussion of the Lie functor $\mathrm{Lie} : \mathbf{LieGrp} \to \mathbf{LieAlg}$.

Theorem 2.46. Let G be a Lie group and let H be a closed subgroup of G. Then H is a regular submanifold of G and hence a Lie group.

The proof of Theorem 2.46 will rely on the following two lemmas.

Lemma 2.47. Let G be a Lie group and let $m : G \times G \to G$ be the product map. Then $D_{(e,e)}m(X,Y) = X + Y$ for all $X, Y \in \mathfrak{g}$.

We have dealt with this already in the proof of Corollary 2.43.

Now, recall that given a Lie group G with Lie algebra $\mathfrak{g}$, a basis of $\mathfrak{g}$ gives rise to exponential coordinates in a neighbourhood of $e \in G$. It will be useful to adapt these kinds of coordinates to other decompositions of $\mathfrak{g}$ as in the following Lemma.

Lemma 2.48. Let G be a Lie group with Lie algebra $\mathfrak{g} = A \oplus B$. Then the map

$$\varphi : \mathfrak{g} \to G,\ \xi \mapsto (\exp_G \pi_A(\xi))(\exp_G \pi_B(\xi)),$$

where $\pi_A : \mathfrak{g} \to A$ and $\pi_B : \mathfrak{g} \to B$ are the canonical projections, has derivative $\mathrm{Id}_{\mathfrak{g}}$ at $0 \in \mathfrak{g}$.

Proof. For all $\xi \in \mathfrak{g}$ we have

$$\begin{aligned}\varphi(\xi) = \exp_G \pi_A(\xi) \exp_G \pi_B(\xi) &= m(\exp_G \pi_A\xi, \exp_G \pi_B\xi)\\ &= m \circ \exp_{G\times G}(\pi_A\xi, \pi_B\xi) = m \circ \exp_{G\times G} \circ(\pi_A \times \pi_B)(\xi).\end{aligned}$$

We compute the derivative of this expression using the chain rule:

$$\begin{aligned}D_0\varphi(\xi) = \underbrace{D_{(e,e)}m}_{+}\underbrace{D_{(0,0)} \exp_{G\times G}}_{\mathrm{id}_{\mathfrak{g}\times\mathfrak{g}}}\underbrace{D_0(\pi_A \times \pi_B)}_{\pi_A\times\pi_B} &= D_{(e,e)}m(\pi_A(\xi), \pi_B(\xi))\\ &= \pi_A(\xi) + \pi_B(\xi) = \xi,\end{aligned}$$

hence the assertion. □

In particular, the map φ in Lemma 2.48 provides us with a chart near the identity by the implicit function theorem.

Proof. (Theorem 2.46). Let $H \leq G$ be a closed subgroup of G and let $\mathfrak{g}$ be the Lie algebra of G. Since H is going to be a regular submanifold, we need to find charts. For this, it is natural to look at $\exp_G^{-1}(H) \subseteq \mathfrak{g}$ which is a closed subset of $\mathfrak{g}$ since H is closed. We think of H as a collection of points in G, possibly accumulating at $e \in G$. If so, we collect the "limiting directions" as follows: Let $\|-\|$ be some norm on $\mathfrak{g}$ and let $S := \{v \in \mathfrak{g} \mid \|v\| = 1\}$. Define $\pi : \mathfrak{g} - \{0\} \to S$ by $v \mapsto v/\|v\|$ and

$$W := \{0\} \cup \left\{ \xi \in \mathfrak{g} - \{0\} \;\middle|\; \begin{array}{l} \exists\ (v_n \in \exp_G^{-1}(H) - \{0\})_{n\in\mathbb{N}} : \\ \lim_n v_n = 0,\ \lim_n \pi(v_n) = \pi(\xi) \end{array} \right\}.$$

Note, that if $\xi \in W$, then also $\lambda\xi \in W$ for all $\lambda \in \mathbb{R}$, hence W is already close to being a vector space. Also, if $W = 0$, then H has to be discrete; so we seem to be on the right track to prove that W has something to do with the Lie algebra of H. Here is another first fact about W:

(i) $\exp_G(W) \subseteq H$.

Certainly, $\exp_G(0) \in H$. Next, let $0 \neq \xi \in W$ and pick a sequence $(v_n)_{n\in\mathbb{N}}$ in H as in the definition. Then

$$\lim_n \frac{v_n}{\|v_n\|} = \frac{\xi}{\|\xi\|} \quad \text{and hence} \quad \xi = \lim_n \frac{\|\xi\|}{\|v_n\|} v_n.$$

Now, let $a_n = \lfloor \|\xi\|/\|v_n\| \rfloor \in \mathbb{N}$ be the integer part of $\|\xi\|/\|v_n\|$. We claim that $\xi = \lim_n a_n v_n$. Indeed,

$$\left\| \frac{\|\xi\|}{\|v_n\|} v_n - a_n v_n \right\| = \left\| \left(\frac{\|\xi\|}{\|v_n\|} - a_n \right) v_n \right\| \leq \|v_n\| \to 0.$$

We hence obtain

$$\exp_G \xi = \lim_n \exp_G(a_n v_n) = \lim_n (\exp_G v_n)^{a_n} \in H$$

by the closedness of H in G.

Moreover, we have the following two facts.

(ii) W is a vector subspace of $\mathfrak{g}$.

We already remarked that $\xi \in W$ implies $\lambda\xi \in W$ for all $\lambda \in \mathbb{R}$ by taking the same sequence $(v_n)_{n\in\mathbb{N}}$. Now let $\xi, \eta \in W$. We want to show that $\xi + \eta \in W$ as well. It suffices to do so under the additional assumption that

ξ, η and $\xi + \eta \in \mathfrak{g}$ are non-zero. By claim (i), homogeneity and the fact that H is a subgroup, we know that $\exp_G t\xi \exp_G t\eta \in H$ for all $t \in \mathbb{R}$. Since $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism on some neighbourhood of $0 \in \mathfrak{g}$, there is an interval $I_\delta = (-\delta, \delta)$ and a smooth curve $u : I_\delta \to \mathfrak{g}$ such that $u(0) = 0$ and

$$\exp_G t\xi \exp_G t\eta = \exp_G u(t)$$

for all $t \in I_\delta$. Taking the derivative at $t = 0$, we deduce from Lemma 2.47 that

$$\left.\frac{d}{dt}\right|_{t=0} \exp_G u(t) = D_0 \exp_G \dot{u}(0) = D_{(e,e)} m(\xi, \eta) = \xi + \eta.$$

$$nv_n = nu(1/n) = \frac{u(1/n)}{1/n} = \frac{u(1/n) - u(0)}{1/n - 0} \xrightarrow{n \to \infty} \xi + \eta.$$

From this we deduce that $v_n \neq 0$ for large n since $\xi + \eta \neq 0$ and that $\lim_n \pi(v_n) = \pi(\xi + \eta)$. Oberserving that $\exp_G v_n \in H$ for all large enough n implies finally that $\xi + \eta \in W$.

(iii) There is an open neighbourhood U of $0 \in \mathfrak{g}$ and a diffeomorphism Φ from U to an open set $\Phi(U) \subseteq G$, such that $\Phi(0) = e$ and $\Phi(U \cap W) = \Phi(U) \cap H$.

Let W' be a vector space complement of $W \subseteq \mathfrak{g}$, then $\mathfrak{g} = W \oplus W'$. From Lemma 2.48 we know that the map

$$\Phi : \mathfrak{g} \to G, \ v \mapsto \exp_G \pi_W(v) \exp_G \pi_{W'}(v)$$

is smooth and a diffeomorphism from an open neighbourhood U of $0 \in \mathfrak{g}$ to the open set $\Phi(U) \subseteq G$. It follows that $\Phi(U \cap W) \subseteq \Phi(U) \cap H$. In order to show that H is a regular submanifold of G it basically suffices to show that in fact $\Phi(U \cap W) = \Phi(U) \cap H$. This supplies us with a coordinate chart at the identity of the desired kind. A coordinate chart at any element may then be obtained by translating around this one.

In order to show equality, we proceed by contradiction: Assume that there is a sequence $(U_n)_{n \in \mathbb{N}}$ of open sets in $\mathfrak{g}$ with the following properties:

(i) $0 \in U_n \ \forall n \in \mathbb{N}$.
(ii) $U_n \subseteq U_{n-1} \subseteq U \ \forall n \in \mathbb{N}_{\geq 2}$.
(iii) $\Phi(U_n \cap W) \subsetneq \Phi(U_n) \cap H$.
(iv) $\bigcap_{n \geq 1} U_n = \{0\}$.

Then from (iii), we deduce the existence of vectors $v_n + v'_n \in U_n$ such that $v_n \in W$ and $v'_n \in W'$ for all $n \in \mathbb{N}$ with $v'_n \neq 0$ and $\exp_G(v_n) \exp_G(v'_n) \in H$. Since $\exp_G v_n \in H$, we deduce that $\exp_G v'_n \in H$ for all $n \in \mathbb{N}$. That is, $v'_n \in \exp_G^{-1}(H) - \{0\}$ for all $n \in \mathbb{N}$. Passing to a subsequence, we may assume that $\lim_n \pi(v'_n)$ exists. Let $\pi(\xi) \in S$ be this limit where $\xi \in \mathfrak{g} - \{0\}$. Then $\xi \in W$. On the other hand,

$$\frac{\xi}{\|\xi\|} = \lim_n \frac{v'_n}{\|v'_n\|} \quad \Rightarrow \quad \xi = \lim_n \frac{\|\xi\|}{\|v'_n\|} v'_n \in W',$$

since subspaces of finite-dimensional vector spaces are closed. Overall, $\xi \in W \cap W' = \{0\}$ which contradicts $\xi \neq 0$.

□

Now that we know that a closed subgroup H of a Lie group G exhibits a strong amount of regularity, namely is a Lie group again, we may also expect that the quotient G/H, which up to now was a locally compact Hausdorff topological space on which G acts continuously, admits some amount of regularity. This is indeed the case.

Theorem 2.49. Let G be a Lie group and let H be a closed subgroup of G. Then there G/H admits the structure of a smooth manifold such that the action map $G \times G/H \to G/H$ is smooth and the canonical projection $\pi : G \to G/H$ is a fibration.

Spaces of the shape G/H as in Theorem 2.49 are called *homogeneous spaces* and constitute some of the most important examples of (Riemannian) manifolds.

Proof. (Idea). Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the Lie algebra of $\mathfrak{h}$ and choose a vector space complement $\mathfrak{k}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Now show that there is $\varepsilon > 0$ such that

$$B_{\mathfrak{k}}(0, \varepsilon) \xrightarrow{\exp_G} G \xrightarrow{\pi} G/H$$

is a homeomorphism of $B_{\mathfrak{k}}(0, \varepsilon)$ with an open neighbourhood of $eH \in G/H$. With a good choice of ε (the same?), you get that the map

$$B_{\mathfrak{k}}(\varepsilon) \times H \to \pi^{-1}(\pi(B_{\mathfrak{k}}(e))), \ (x, h) \mapsto \exp(x)h$$

is a diffeomorphism which implies that π is a fibration. □

The fact that the map $\pi : G \to G/H$ is a fibration, implies for instance that it admits locally a smooth section. Fibrations are useful for instance in algebraic topology in order to compute e.g. fundamental groups of G in terms of the fundamental groups of H and G/H.

2.6. **The Adjoint Representation.** Next to the exponential map of a Lie group, the *adjoint representation* is the second most fundamental tool to study Lie groups: Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the map $\mathrm{int}(g) : G \to G$, $x \mapsto gxg^{-1}$ is a smooth automorphism of G and the associated map $\mathrm{int} : G \to \mathrm{Aut}(G)$, $g \mapsto \mathrm{int}(g)$ is a homomorphism.

For every $g \in G$, let $\mathrm{Ad}(g) := D_e\mathrm{int}(g) : \mathfrak{g} \to \mathfrak{g}$. Then $\mathrm{Ad}(g)$ is in fact an element of $\mathrm{GL}(\mathfrak{g})$: This follows from the fact, that the map $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ is a homomorphism, $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2)$ which in turn follows from the chain rule and the fact that int is a homomorphism. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is called the *adjoint representation* of G.

In general, representations allow one to analyze the group structure using all one's linear algebra knowledge and for this reason turn out to be a powerful tool.

Now, applying Corollary 2.40 to $\mathrm{int}(g) : G \to G$, we have

$$g \exp_G(tX) g^{-1} = \exp_G(t \,\mathrm{Ad}(g)X)$$

for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Let us see what this means for $G = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_{n,n}(\mathbb{R})$ and

$$g \left(\sum_{n=0}^{\infty} \frac{t^n X^n}{n!} \right) g^{-1} = g \,\mathrm{Exp}(tX) g^{-1} = \mathrm{Exp}(t \,\mathrm{Ad}(g)X) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (gXg^{-1})^n$$

and hence

$$\mathrm{Ad}(g)X = gXg^{-1}$$

by comparing coefficients.

Definition 2.50. Let G be a Lie group. A *representation* of G into a (real or complex) finite-dimensional vector space V is a smooth homomorphism $\pi : G \to \mathrm{GL}(V)$.

We will denote by $\mathfrak{gl}(V)$ the Lie algebra of the Lie group $\mathrm{GL}(V)$, as a vector space it is just $\mathrm{End}(V)$. The Lie algebra analogue of Definition 2.50 is the following.

Definition 2.51. Let $\mathfrak{g}$ be a Lie algebra. A *representation* of $\mathfrak{g}$ into a (real or complex) finite-dimensional vector space V is a Lie algebra homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$.

A fundamental example to Definition 2.50 is the adjoint representation Ad. For definition 2.51, we define

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \ \ X \mapsto [X, -]$$

which we also call the adjoint representation. It is in fact a representation of the Lie algebra $\mathfrak{g}$ as

$$\begin{aligned}\mathrm{ad}([X,Y])(Z) &= [[X,Y],Z] = -[[Y,Z],X] - [[Z,X],Y] = [X,[Y,Z]] - [Y,[X,Z]] \\ &= \mathrm{ad}(X)\circ\mathrm{ad}(Y)(Z) - \mathrm{ad}(Y)\circ\mathrm{ad}(X)(Z) = [\mathrm{ad}(X),\mathrm{ad}(Y)](Z)\end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$.

It is an important theorem of Ado that every Lie algebra admits a faithful representation. The proof of this theorem starts out with ad and then deals with its kernel.

For now, we establish the following important link between $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Proposition 2.52. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The derivative of Ad at $e \in G$ is ad.

Proof. By definition, we have for $h \in G$ and $Y \in \mathfrak{g}$: $\mathrm{Ad}(h)(Y) = D_e\mathrm{int}(h)(Y)$. Considering the left-invariant vector field $\mathrm{Ad}(h)(Y)^L$ associated to $\mathrm{Ad}(h)(Y)$ we have for all $f \in C^\infty(G)$ and $g \in G$:

$$\mathrm{Ad}(h)(Y)^L f(g) = \frac{d}{ds}\Big|_{s=0} f(gh\exp(sY)h^{-1})$$

Putting $h = \exp tX$ for some $t \in \mathbb{R}$ and $X \in \mathfrak{g}$ we obtain

$$\mathrm{Ad}(\exp tX)(Y)^L f(g) = \frac{d}{ds}\Big|_{s=0} f(g\exp tX \exp sY \exp -tX).$$

On the other hand, by Corollary 2.40, we have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "D_e\,\mathrm{Ad}"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\mathrm{Exp}"'] \end{tikzcd}$$

and hence

$$\begin{aligned}\mathrm{Ad}(\exp tX) = \mathrm{Exp}(D_e\,\mathrm{Ad}(tX)) &= I + tD_e\,\mathrm{Ad}(X) + \frac{t^2}{2!}(D_e\,\mathrm{Ad}(X))^2 + \cdots \\ &= I + tD_e\,\mathrm{Ad}(X) + t^2 v(t,X)\end{aligned}$$

where $v(t,X)$ is a smooth function in t. Evaluating on Y, forming the associated left-invariant vector field and acting on a function $f \in C^\infty(G)$ yields for $g \in G$:

$$\mathrm{Ad}(\exp(tX))(Y)^L f(g) = Y^L f(g) + tD_e\,\mathrm{Ad}(X)(Y)^L f(g) + t^2 v(t,X)(Y)^L f(g)$$

Taking the derivative $d/dt|_{t=0}$ of our two expressions of $\mathrm{Ad}(\exp(tX))(Y)^L f(g)$ yields

$$\begin{aligned}
D_e \mathrm{Ad}(X)(Y)^L f(g) &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(g \exp tX \exp sY \exp -tX) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(g \exp tX \exp sY) - \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(g \exp sY \exp tX) \\
&= \frac{d}{dt}\Big|_{t=0} Y^L f(g \exp tX) - \frac{d}{ds}\Big|_{s=0} X^L f(g \exp sY) \\
&= X^L Y^L f(g) - Y^L X^L f(g) \\
&= [X^L, Y^L] f(g)
\end{aligned}$$

□

Here is a shorter proof of Proposition 2.52 assuming the Lie derivative from differential geometry.

Proof. By Proposition 2.40 we have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \\
\mathfrak{g} \arrow[u, "\exp"] \arrow[r, "D_e \mathrm{Ad}"] & \mathfrak{gl}(\mathfrak{g}), \arrow[u, "\mathrm{Exp}"']
\end{tikzcd}$$

hence for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$:

$$\mathrm{Exp}(D_e \mathrm{Ad}(tX)) = \mathrm{Ad}(\exp(tX)).$$

The exponential map on the left hand side of the above equation is the power series exponential. Hence for $X, Y \in \mathfrak{g}$ and $t \in \mathbb{R}$:

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Exp}(D_e \mathrm{Ad}(tX))(Y) = D_e \mathrm{Ad}(X)(Y)$$

which is what we are trying to determine. The derivative of the right hand side at $t = 0$ can be manipulated as follows:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))(Y) &= \frac{d}{dt}\Big|_{t=0} D_e \mathrm{int}(\exp(tX))(Y) \\
&= \frac{d}{dt}\Big|_{t=0} D_e(R_{\exp(-tX)} \circ L_{\exp(tX)})(Y) \\
&= \frac{d}{dt}\Big|_{t=0} D_{\exp(tX)} R_{\exp(-tX)} D_e L_{\exp(tX)}(Y) \\
&= \frac{d}{dt}\Big|_{t=0} D_{\exp(tX)} R_{\exp(-tX)} (Y^L_{\exp(tX)}) \\
&= \frac{d}{dt}\Big|_{t=0} D_{\Phi^X_t(e)} \Phi^X_{-t} (Y^L_{\Phi^X_t(e)}) \\
&= [X, Y]
\end{aligned}$$

where the last equality is assumed to be known from differential geometry. □

To study the structure of Lie groups, one studies closed normal subgroups, i.e. closed subgroups that are invariant under conjugation. That is why the adjoint representations and Proposition 2.52 will be very important. For instance, we have the following consequences of the above discussion.

Corollary 2.53. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is abelian then so is $\mathfrak{g}$. Conversely, if $\mathfrak{g}$ is abelian and G is connected, then G is abelian.

Proof. If G is abelian, then $\mathrm{int}(g) = \mathrm{id}_G$ for all $g \in G$ and hence $\mathrm{Ad}(g) = D_e\mathrm{int}(g) = \mathrm{id}_{\mathfrak{g}}$ which implies that $\mathrm{ad}(X) = D_e \mathrm{Ad}(X) \equiv 0$. That is, $\mathfrak{g}$ is abelian. Conversely, if $\mathfrak{g}$ is abelian, then a small neighbourhood of $e \in G$ is abelian. However, if G is connected, that neighbourhood generates G. □

Definition 2.54. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A subgroup $N \leq G$ is *normal* in G, denoted $N \trianglelefteq G$, if $gNg^{-1} \subseteq N$ for all $g \in G$. Recall that in this case, one has a natural group structure on G/N given by $gN \cdot hN := ghN$.

A subalgebra $\mathfrak{n} \leq \mathfrak{g}$ is an *ideal*, denoted $\mathfrak{n} \trianglelefteq \mathfrak{g}$, if $[X, \mathfrak{n}] \subseteq \mathfrak{n}$ for all $X \in \mathfrak{g}$.

Proposition 2.55. Let G be Lie group with Lie algebra $\mathfrak{g}$. Further, let N be a closed subgroup of G with Lie algebra $\mathfrak{n} \leq \mathfrak{g}$. If $N \trianglelefteq G$, then $\mathfrak{n} \trianglelefteq \mathfrak{g}$. Conversely, if $\mathfrak{n} \trianglelefteq \mathfrak{g}$ and N and G are connected, then $N \trianglelefteq G$.

Proof. For $g \in G$ and $Y \in \mathfrak{n}$ we have $g \exp_G tY g^{-1} = \exp_G(t \,\mathrm{Ad}(g)(Y))$ for all $t \in \mathbb{R}$. Hence, if $N \trianglelefteq G$, then $\exp_G(t \,\mathrm{Ad}(g)(Y)) \in N$ for all $t \in \mathbb{R}$ and therefore $\mathrm{Ad}(g)(Y) \in \mathfrak{n}$. If we let $g := \exp tX$ for some $X \in \mathfrak{g}$ so that $\mathrm{Ad}(\exp_G tX)(Y) \in \mathfrak{n}$ then by taking the derivative $d/dt|_{t=0}$ yields $\mathrm{ad}(X)(Y) \in \mathfrak{n}$ which shows that $\mathfrak{n}$ is an ideal.

It is left as an exercise to reverse the process if G and N are connected. □

Corollary 2.56. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Further, let $N \trianglelefteq G$ be a closed normal subgroup of G with Lie algebra $\mathfrak{n} \leq \mathfrak{g}$. Then the manifold structure on the quotient group G/N introduced in Theorem 2.49 turns G/N into a Lie group whose Lie algebra naturally identifies with $\mathfrak{g} \,/\, \mathfrak{n}$.

3. Structure Theory

This chapter deals with the general structure theory of Lie groups in three major parts: The first looks at the important notion of *solvable* Lie groups and Lie algebras, the definition of which is the same as in finite group theory. And these Lie groups are as prominent as finite solvable groups are in Galois Theory. However, whereas finite solvable groups are very hard to understand, every solvable Lie group is well-understood from the groups of upper-triangular matrices.

Moreover, for a general Lie group G, we will define its radical $\mathrm{rad}(G)$ to be the unique maximal solvable normal subgroup of G. The quotient $G/\mathrm{rad}(G)$ is then going to be a *semisimple* Lie group which can be decomposed into *simple* Lie groups which in turn are completely classifiable. Semisimple Lie groups and Lie algebras are the content of the third part of this chapter. The second deals with nilpotent Lie groups and Lie algebras. Nilpotency is a strengthening of solvability just as in finite group theory.

3.1. **Solvable Lie Groups and Lie Algebras.**

Definition 3.1. A group G is *solvable* if there is a sequence of subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

such that the quotients G_{i-1}/G_i for $i \in \{1, \dots n\}$ are abelian.

Hence a solvable group is one that arises from the trivial group by a sequence of extensions by abelian groups. There is another description of solvable groups in terms of commutator series: Let $G^{(1)} := [G, G]$ be the subgroup of G which is generated by the set $\{[x, y] \mid x, y \in G\}$ where $[x, y] = xyx^{-1}y^{-1}$ (there will be no reason to confuse this notation with a Lie bracket). Clearly, $G^{(1)}$ is normal in G, in fact it is characteristic in G, i.e. invariant under all automorphisms of G. Furthermore, given any homomorphism $\varphi : G \to A$ of G to an abelian group A, we have $\varphi([x, y]) = [\varphi(x), \varphi(y)] = e$ and hence $\ker \varphi \supseteq G^{(1)}$. Therefore, $G/G^{(1)}$ is the largest abelian quotient of G.

Definition 3.2. Let G be a group. The *derived series* of G is defined inductively by $G^{(1)} := [G, G]$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $i \in \mathbb{N}_{\geq 2}$.

It is immediate that $G^{(i-1)}/G^{(i)}$ is abelian. Also, by a recurrence argument, one sees that $G^{(i)}$ is in fact a characteristic subgroup of G.

Lemma 3.3. Let G be a group. Then G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 1$.

Definition 3.4. Let G be a solvable group. The *solvability length* $\mathrm{sol}(G)$ of G is defined by

$$\mathrm{sol}(G) := \min\{n \in \mathbb{N} \mid G^{(n)} = e\}.$$

Proof. (Lemma 3.3). Clearly, if $G^{(n)} = \{e\}$ for some $n \in \mathbb{N}$, then the sequence

$$G \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots \trianglerighteq G^{(n)} = \{e\}$$

shows that G is solvable. Conversely, if G is solvable, there is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n \trianglerighteq \{e\}$$

with abelian quotients. For instance, G/G_1 is abelian and hence $\ker(G \to G/G_1) \supseteq G^{(1)} = [G, G]$, that is: $G_1 \supseteq G^{(1)}$. By recurrence, we get $G_i \supseteq G^{(i)}$ for all $i \in \{1, \dots, n\}$ and hence $G^{(n)} = \{e\}$. $\square$

The following lemma shows how solvability behaves with respect to subgroups and quotients.

Lemma 3.5.

(i) Let G be a group and let H be a subgroup of G. If G is solvable, so is H.

(ii) Let $\{e\} \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to \{e\}$ be an exact sequence of groups; that is, i is an injective homomorphism, π is a surjective homomorphisms and $\operatorname{im} i = \ker \pi$. Then G is solvable if and only if N and Q are solvable.

Proof. As to (i), we certainly have $H^{(i)} \subseteq G^{(i)}$ for all $i \in \mathbb{N}$, hence the assertion follows from Lemma 3.3. For (ii), assume that G is solvable. Then N is solvable as a subgroup of G by (i). Further, note that by surjectivity of π we have $Q^{(i)} = \pi(G^{(i)})$ and hence $Q^{(n)} = \{e\}$ if $G^{(n)} = \{e\}$. Conversely, assume that N and Q are solvable. Let $r = \mathrm{sol}(Q)$. Then $\pi(G^{(r)}) = Q^{(r)} = \{e\}$ implies $G^{(r)} \subseteq \operatorname{im} i$. Hence, if $m = \mathrm{sol}(N)$, we have $G^{(r)(m)} \subseteq i(N^{(m)}) = \{e\}$. Observing $G^{(r)(m)} = G^{(r+m)}$ thus implies that G is solvable of length at most $r + m$. □

When dealing with topological groups, it is desirable to build a solvable group from *Hausdorff* abelian groups, hence the groups that occur in the according sequence should be closed. It is an important observation that this is always possible.

Theorem 3.6. Let G be a T_1 topological group (which hence is T_2). Then G is solvable if and only if there is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

such that G_i is closed in G for all $i \in \{1, \dots, n\}$ and G_{i-1}/G_i is abelian for all $i \in \{1, \dots n\}$.

A key observation towards the proof is that if G is a T_1 topological group and H is an abelian subgroup of G then $\overline{H}$ is abelian as well, by continuity of the map $G \times G \to G,\ (x, y) \mapsto [x, y]$.

Proof. Clearly, the condition is sufficient. To show that it is necessary, we proceed by recurrence on the solvability length $\mathrm{sol}(G)$ of G. If $\mathrm{sol}(G) = 1$, then G is abelian and $G_1 := \{e\}$ serves because it is closed by the T_1 assumption. Also, $G/G_1 \cong G$ is abelian as desired. Now, assume $n := \mathrm{sol}(G) \geq 2$ and that the theorem holds for all groups of solvability length at most $n - 1$. Since $\mathrm{sol}(G) = n$, we have $\{e\} = G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \cdots \trianglelefteq G_0 = G$. Also, $G^{(n-1)}$ is normal in G and abelian. Hence so is $\overline{G^{(n-1)}}$ and therefore the quotient $G/\overline{G^{(n-1)}}$ is a T_1 topological group which in turn is a quotient of $G/G^{(n-1)}$. Thus $G/\overline{G^{(n-1)}}$ has solvability length at most $n-1$. By the induction hypothesis, we therefore have a sequence

$$\{\overline{e}\} = H_{n-1} \trianglelefteq H_{n-2} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = H := G/\overline{G^{(n-1)}}$$

where H_i is closed in H and H_{i-1}/H_i is abelian for all $i \in \{1, \dots, n-1\}$. Consider now the canonical projection $\pi : G \to G/\overline{G^{(n-1)}}$ and define $G_i := \pi^{-1}(H_i)$ for all $i \in \{1, \dots, n-1\}$ as well as $G_n := \{e\}$. Then each G_i is closed in G since π is continuous and $G_i \trianglelefteq G_{i-1}$ since π. Finally, by surjectivity of π, $G_{i-1}/G_i \cong H_{i-1}/H_i$ which is abelian. □

Lemma 3.7. Let G be a connected group. Then $G^{(i)}$ is connected for all $i \in \mathbb{N}$.

Proof. Since $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, it suffices to show that $G^{(1)} = [G, G]$ is connected. Recall, that $G^{(1)}$ is the subgroup of G generated by the set $V = \{[x, y] \mid x, y \in G\}$. Since V is symmetric, this implies $G^{(1)} = \bigcup_{n \geq 1} V^n$. Now observe that V is the image of the continuous map $G \times G \to G,\ (x, y) \mapsto [x, y]$ and hence is connected as G is connected. Similarly, V^n is the image of a continous map $V \times V \times \cdots \times V \to G$ and hence is connected. In addition, we have $e \in V^n$ for all $n \in \mathbb{N}$ and hence $\bigcup_{n \geq 1} V^n$ is connected by point-set topology. □

The following is a slight improvement of Theorem 3.6

Corollary 3.8. Let G be a T_1 connected topological group. Then G is solvable if and only if there is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

such that G_i is closed, connected and normal in G for all $i \in \{1, \dots, n\}$ and G_{i-1}/G_i is abelian for all $i \in \{1, \dots n\}$.

Proof. By Theorem 3.6 there is a sequence

$$G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$$

such that H_i is closed in G for all $i \in \{1, \dots, n\}$ and H_{i-1}/H_i is abelian for all $i \in \{1, \dots n\}$. Now, since G/H_1 is abelian, we have $G^{(1)} \subseteq H_1$; and since H_1 is closed, we even have $\overline{G^{(1)}} \subseteq H_1$. Therefore, we set $G_1 := \overline{G^{(1)}}$ which is connected by Lemma 3.7. Now just continue in this fashion with $G_i := \overline{G^{(i)}}$ for all $i \in \{1, \dots, n\}$. $\square$

Here are examples of solvable groups which, in a sense to be made clear, are universal.

Example 3.9. Let k be a field, $n \in \mathbb{N}$ and consider the group

$$S := \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{(n-1)n} \\ & & & x_{nn} \end{pmatrix} \middle| \begin{array}{l} \det(x_{ij})_{i,j} = \prod_{i=1}^n x_{ii} \neq 0, \\ \forall\ 1 \leq j < i \leq n:\ x_{ij} = 0 \end{array} \right\} \subseteq \mathrm{GL}(n,k)$$

A direct calculation then shows

$$S^{(1)} \subseteq \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \right\} \quad , \quad S^{(2)} \subseteq \left\{ \begin{pmatrix} 1 & 0 & * & \cdots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix} \right\}$$

and eventually $S^{(n)} = \{\mathrm{Id}_n\}$, hence S is solvable. For $k = \mathbb{R}$, the group S satisfies $S/S^{(1)} \cong \mathbb{R}^{*n}$. Since $S^{(1)}$ is (path-)connected, this implies that S has 2^n connected components.

Here is an important corollary of the so far discussion applied to solvable *Lie* groups.

Corollary 3.10. Let G be a connected Lie group. Then G is solvable if and only if there is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

such that G_i is connected and closed in G for all $i \in \{1, \dots, n\}$ and G_{i-1}/G_i is isomorphic to either S^1 or $\mathbb{R}$ for all $i \in \{1, \dots n\}$.

Note that S^1 and $\mathbb{R}$ are exactly the 1-dimensional connected abelian Lie groups which is a consequence of the following remark: Recall from Corollary 2.43 and the subsequent discussion that if L is a connected abelian Lie group with Lie algebra $\mathfrak{L}$ then $\exp : \mathfrak{L} \to L$ is a surjective homomorphism with discrete kernel Γ such that $L \cong \mathbb{R}^n / \Gamma \cong (S^1)^r \times \mathbb{R}^{n-r}$ if $n = \dim \mathfrak{L} = \dim L$. More precisely, one can show by recurrence that if $V = \langle \Gamma \rangle \leq \mathfrak{L}$ then there is a basis $(v_1, \dots, v_r)$ of V consisting of elements of Γ such that $\Gamma = \mathbb{Z}\, v_1 + \cdots + \mathbb{Z}\, v_r$. Complete this to a basis $(v_1, \dots, v_r, v_{r+1}, \dots, v_n)$ of $\mathfrak{L}$ and conclude $L \cong \mathfrak{L} / \Gamma \cong (S^1)^r \times \mathbb{R}^{n-r}$. This helps with Corollary 3.10 as follows.

Proof. (Corollary 3.10). Let

$$G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$$

be a sequence as in Corollary 3.8. Since G is a Lie group and the H_i ($i \in \{0, \dots n\}$) are closed in G, they are Lie groups as well. Hence H_{i-1}/H_i is a connected abelian Lie group for all $i \in \{1, \dots n\}$. By the above discussion we therefore have $H_{i-1}/H_i \cong (S^1)^{a_i} \times \mathbb{R}^{b_i}$ for some $a_i, b_i \in \mathbb{N}_0$. If we denote $(S^1)^{a_i} \times \mathbb{R}^{b_i}$ by $L(a_i, b_i)$ we have for each $i \in \{1, \dots, n\}$ the following chain of subgroups of $H_{i-1}/H_i = L(a_i, b_i)$:

$$L(a_i, b_i) \geq L(a_i - 1, b_i) \geq \cdots \geq L(0, b_i) \geq L(0, b_i - 1) \geq \cdots \geq L(0,0) = \{e\}.$$

Taking the inverse image of this sequence under the projection $H_{i-1} \to H_{i-1}/H_i$ for each $i \in \{1, \dots, n\}$ refines the sequence we started with as desired. □

Having discussed the notion of solvability for (Lie) groups, we now turn to the analogous notion for Lie algebras: Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^{(1)}$ be the vector subspace of $\mathfrak{g}$ generated by $\{[X, Y] \mid X, Y \in \mathfrak{g}\}$. Then $\mathfrak{g}^{(1)}$ is an ideal in $\mathfrak{g}$, the quotient $\mathfrak{g} \,/\, \mathfrak{g}^{(1)}$ is abelian and given any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{n}$ to an abelian Lie algebra $\mathfrak{n}$, we have $\ker \varphi \supseteq \mathfrak{g}^{(1)}$. Therefore, $\mathfrak{g} \,/\, \mathfrak{g}^{(1)}$ is the largest abelian quotient of $\mathfrak{g}$. The proof of these assertions is formal. For instance, to show that $\mathfrak{g}^{(1)}$ is an ideal in $\mathfrak{g}$, let $Z \in \mathfrak{g}$ and $A \in \mathfrak{g}^{(1)}$. Then by definition, $A = \sum_{i=1}^{n} \lambda_i [X_i, Y_i]$ for some $\lambda_i \in \mathbb{R}$ and $X_i, Y_i \in \mathfrak{g}$. Then compute, using bilinearity of the bracket, that

$$[Z, A] = \sum_{i=1}^{n} \lambda_i [Z, [X_i, Y_i]]$$

which is a linear combination of brackets in $\mathfrak{g}$, hence contained in $\mathfrak{g}^{(1)}$. Overall, this shows that $\mathfrak{g}^{(1)}$ is invariant under left-multiplication with elements of $\mathfrak{g}$. The remaining assertions are equally simple to prove.

Definition 3.11. Let $\mathfrak{g}$ be a Lie algebra. The derived series of $\mathfrak{g}$ is defined inductively by $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(i)} := [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \in \mathbb{N}_{n \geq 2}$.

Definition 3.12. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$. The smallest such n is called the *solvability length* $\operatorname{sol}(\mathfrak{g})$ of $\mathfrak{g}$.

Lemma 3.13. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the following are equivalent:

(i) The Lie algebra $\mathfrak{g}$ is solvable.

(ii) There is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \mathfrak{g}_2 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_n = 0$$

of subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_{i-1} \,/\, \mathfrak{g}_i$ is abelian for all $i \in \{1, \dots, n\}$.

(iii) There is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \mathfrak{g}_2 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_n = 0$$

of subalgebras of $\mathfrak{g}$ such that $\dim \mathfrak{g}_{i-1} \,/\, \mathfrak{g}_i = 1$ is one-dimensional for all $i \in \{1, \dots, n\}$.

Proof. To see that (i) implies (ii), just take the derived series. Clearly, also (ii) implies (i). For (ii) implies (iii), choose a basis $(v_1, \dots, v_{j_i})$ of $\mathfrak{g}_{i-1} \,/\, \mathfrak{g}_i$ for every $i \in \{1, \dots, n\}$ and take the inverse image of the sequence

$$0 \leq \langle v_1 \rangle \leq \langle v_1, v_2 \rangle \leq \cdots \leq \langle v_1, \dots, v_{j_i} \rangle = \mathfrak{g}_{i-1} \,/\, \mathfrak{g}_i$$

under the canonical projection $\mathfrak{g}_{i-1} \to \mathfrak{g}_{i-1} \,/\, \mathfrak{g}_i$ to refine the original sequence at each step to obtain one-dimensional quotients.

Eventually, to see that (iii) implies (ii), just note that one-dimensional Lie algebras are necessarily abelian: If $\mathfrak{a} = \langle e \rangle$, then for all $\lambda, \mu \in \mathbb{R}$ we have

$$\langle \lambda e, \mu e \rangle = \lambda\mu[e,e] = -\lambda\mu[e,e] = -[\lambda e, \mu e] \quad \Rightarrow \quad [\lambda e, \mu e] = 0,$$

i.e. $\mathfrak{a}$ is abelian. □

As in the group case, there is the following easy-to-prove statement about subalgebras and extensions.

Proposition 3.14.

(i) Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable, then so is $\mathfrak{h}$.

(ii) Let $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ be an exact sequence of Lie algebras. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{n}$ and $\mathfrak{q}$ are solvable. In this case, $\mathrm{sol}(\mathfrak{g}) \leq \mathrm{sol}(\mathfrak{n}) + \mathrm{sol}(\mathfrak{q})$.

We now establish the relationship between solvability of Lie groups and solvability of Lie algebras.

Theorem 3.15. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the following statements hold.

(i) If G is solvable, then $\mathfrak{g}$ is solvable.

(ii) If $\mathfrak{g}$ is solvable and G is connected, then G is solvable.

Note that we already have the above theorem with "solvable" replaced by "abelian" (Corollary 2.53). This is going to be the induction basis. Also, we shall need to make use of the following lemma.

Lemma 3.16. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal in $\mathfrak{g}$. Then $N := \langle \exp X \mid X \in \mathfrak{n} \rangle$ is a normal subgroup of G.

This is of course going to come from the adjoint representations.

Proof. Since $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ we have $\mathrm{ad}(X)(\mathfrak{n}) \subseteq \mathfrak{n}$ for all $X \in \mathfrak{g}$. Hence also $\exp(\mathrm{ad}(X)(\mathfrak{n})) \subseteq N$ for all $X \in \mathfrak{g}$ and therefore $\mathrm{Ad}(\exp X)(\mathfrak{n}) \subseteq \mathfrak{n}$ for all $X \in \mathfrak{g}$; this implies $\mathrm{Ad}(g)(\mathfrak{n}) \subseteq \mathfrak{n}$ for all $g \in G$ by connectedness of G. Finally, since $g \exp Y g^{-1} = \exp(\mathrm{Ad}(g)(Y))$, we deduce that the set $\{\exp Y \mid Y \in \mathfrak{n}\}$ is invariant under G-conjugation, i.e. N is normal in G. □

We are now able to prove the theorem.

Proof. (Theorem 3.15). For (i), note that solvability of G implies solvability of G_0, the connected component of the identity, which has the same Lie algebra as G. Now apply Corollary 3.10 to obtain a sequence

$$G_0 = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$$

in which each H_i ($i \in \{1, \ldots, n\}$) is closed and connected in G and H_{i-1}/H_i ($i \in \{1, \ldots, n\}$) is isomorphic to either S^1 or $\mathbb{R}$. Now, let $\mathfrak{h}_i := \mathrm{Lie}(H_i)$. Then we have a sequence

$$\mathfrak{g} = \mathfrak{h}_0 \trianglerighteq \mathfrak{h}_1 \trianglerighteq \cdots \trianglerighteq \mathfrak{h}_n = 0.$$

Since $\mathfrak{h}_{i-1} / \mathfrak{h}_i \cong \mathrm{Lie}(H_{i-1}/H_i)$ ($i \in \{1, \ldots, n\}$) we have $\dim(\mathfrak{h}_{i-1} / \mathfrak{h}_i) = 1$ and hence each $\mathfrak{h}_{i-1} / \mathfrak{h}_i$ is abelian. Therefore $\mathfrak{g}$ is solvable.

For (ii), we argue by recurrence on the solvability length $\mathrm{sol}(\mathfrak{g})$ of $\mathfrak{g}$. By Corollary 2.53, we already have the assertion for $\mathrm{sol}(\mathfrak{g}) = 1$. Now assume that $n := \mathrm{sol}(\mathfrak{g}) \geq 2$ and let $\mathfrak{g} \trianglerighteq \mathfrak{g}^{(1)} \trianglerighteq \cdots \trianglerighteq \mathfrak{g}^{(n-1)} \trianglerighteq \mathfrak{g}^{(n)} = 0$ be the derived series of $\mathfrak{g}$. Observe that $\mathfrak{g} / \mathfrak{g}^{(n-1)}$ is solvable of length $n-1$. Also, $\mathfrak{g}^{(n-1)}$ is abelian and hence by 2.43, the exponential map $\exp : \mathfrak{g}^{(n-1)} \to G$ is a homomorphism. Therefore, $H := \overline{\exp(\mathfrak{g}^{(n-1)})}$ is a closed abelian subgroup of G which by the preceeding Lemma 3.16 is normal in G. Let $\mathfrak{h} := \mathrm{Lie}(H)$. Then $\mathfrak{h} \trianglelefteq \mathfrak{g}$ and $\mathfrak{g}^{(n-1)} \subseteq \mathfrak{h}$. Hence $\mathfrak{g} / \mathfrak{h}$, which is the Lie algebra of the connected Lie group G/H, is a quotient of $\mathfrak{g}\,/\,\mathfrak{g}^{(n-1)}$ and hence of solvability length at most $n-1$. Therefore, by recurrence, G/H is solvable. Also, H is solvable since it is abelian. Hence G is solvable by Lemma 3.5. □

Note our constant struggle to pass between properties of Lie groups and Lie algebras. For instance, to show that solvability of $\mathfrak{g}$ implies solvability of G one cannot simply exponentiate commutators. This tracks back to a lack of knowledge of the inverse of the exponential map in a small neighbourhood of the identity which is given by the Baker-Campbell-Hausdorff formula; we might come back to this.

But now, we prove the first real structure theorem of Lie groups, more specifically, closed connected solvable subgroups of $\mathrm{GL}(n,\mathbb{C})$. To this end, we will widen the scope and consider smooth homomorphisms $\varrho : G \to \mathrm{GL}(V)$ where G is a connected solvable Lie group, V is a finite-dimensional $\mathbb{C}$-vector space and $\mathrm{GL}(V)$ is considered as a real Lie group.

We refrain from using the isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}(\dim V, \mathbb{C})$ because the structure theorem precisely consists in providing a particularly nice basis. Also, we use $\mathbb{C}$-vector spaces V to make sure that every endomorphism of V has at least one eigenvector.

As nicely as the structure theorem will deal with this situation, it does not cover all connected solvable Lie groups.

Example 3.17. Consider the Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} \subseteq \mathrm{GL}(3, \mathbb{R})$$

and the subgroup

$$\Gamma := \left\{ \begin{pmatrix} 1 & 0 & m \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| \, m \in \mathbb{Z} \right\}$$

which is contained in the center of G and hence is normal in G. Then the quotient G/Γ does not admit any smooth injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

Definition 3.18. Let G be a Lie group and let (ϱ, V) be a smooth, finite-dimensional, complex representation of G. A *weight vector* of G in (ϱ, V) is a non-zero vector $v_0 \in V$ which is an eigenvector of $\varrho(g)$ for all $g \in G$. Then $\varrho(g)v_0 = \chi(g)v_0$ for every $g \in G$ where $\chi : G \to \mathbb{C}^*$ is a smooth homomorphism.

Conversely, given any smooth homomorphism $\chi : G \to \mathbb{C}^*$, define $V_\chi := \{v \in V \mid \varrho(g)v = \chi(g)v \ \forall g \in G\}$. Then χ is a *weight* of G in (ϱ, V) if $V_\chi \neq 0$.

In this terminology, Lie's structure theorem reads as follows.

Theorem 3.19. Let G be a connected, solvable Lie group and let (ϱ, V) be a smooth, finite-dimensional, complex representation of G. If $V \neq 0$, then G has a weight in (ϱ, V).

By recurrence, this theorem implies the following corollary.

Corollary 3.20. Let G be a connected, solvable Lie group and let (ϱ, V) be a smooth, finite-dimensional, complex representation of G. Then there is a basis of V such that each $\varrho(g)$ is upper-triangular with respect to this basis.

The proof of Theorem 3.19 will rely on the following two lemmas.

Lemma 3.21. Let G be a Lie group and let (ϱ, V) be a smooth, finite-dimensional, real or complex representation of G. Then the following statements hold.

(i) If $W \le V$ is $\varrho(G)$-invariant, then W is $D_e\varrho(\mathfrak{g})$-invariant.
(ii) If $W \le V$ is $D_e\varrho(\mathfrak{g})$-invariant and G is connected, then W is $\varrho(G)$-invariant.

Proof. This follows from $\varrho(\exp tX) = \operatorname{Exp}(tD_e\varrho(X)) \ \forall\, t \in \mathbb{R},\ X \in \mathfrak{g}$. □

The second lemma is key and uses a particularly nice argument.

Lemma 3.22. Let G be a connected Lie group and let (ϱ, V) be a smooth, finite-dimensional, complex representation of G. Let $H \trianglelefteq G$ be a normal subgroup of G and let $\chi : H \to \mathbb{C}^*$ be a weight of H in $(\varrho|_H, V)$. Then $V_\chi = \{v \in V \mid \varrho(h)v = \chi(h)v\ \forall h \in H\}$ is $\varrho(G)$-invariant.

Proof. Let $g \in G$ and let $v \in V_\chi$. Then for all $h \in H$, we have

$$\varrho(h)\varrho(g)v = \varrho(g)\varrho(g^{-1}hg)v = \varrho(g)\chi(g^{-1}hg)v = \chi(g^{-1}hg)\varrho(g)v$$

and it remains to show that $\chi(g^{-1}hg) = \chi(h)$ for all $g \in G$. Notice that by the above, $\chi(g^{-1}hg)$ is an element of the spectrum $\operatorname{Spec}(\varrho(h))$ of $\varrho(h)$, which is the set of eigenvalues of $\varrho(h)$ and which by finite-dimensionality inherits the discrete topology from $\mathbb{C}$. Then the map $G \to \operatorname{Spec}(\varrho(h)),\ g \mapsto \chi(g^{-1}hg)$ is continuous (for instance, the inverse image of a closed set is closed) and hence constant equal to, say, $\chi(e^{-1}he) = \chi(h)$, since G is connected. □

We are now able to prove Lie's Theorem 3.19.

Proof. Let G and (ϱ, V) be as stated in the theorem. We proceed by recurrence on the dimension $\dim(\mathfrak{g})$ of $\mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \langle X \rangle$. Let $v_0 \in V$ be an eigenvector of $D_e\varrho(X)$ (which exists by the fundamental theorem of algebra). Then letting $W := \langle v_0 \rangle$, we have $D_e\varrho(X)(W) \subseteq W$ and hence $D_e\varrho(\mathfrak{g})(W) \subseteq W$ which by Lemma 3.21 implies that $\varrho(G)W \subseteq W$, hence v_0 is a weight vector.

If $\dim \mathfrak{g} \ge 2$, we choose a closed, connected, normal subgroup $G_1 \lhd G$ by Corollary 3.10 such that G/G_1 is isomorphic to either S^1 or $\mathbb{R}$. Let $\mathfrak{g}_1$ be the ideal in $\mathfrak{g}$ corresponding to G_1. Clearly, $\mathfrak{g} = \mathfrak{g}_1 + \langle Y \rangle$ where Y is a fixed choice of vector not in $\mathfrak{g}_1$. Since G_1 is connected, solvable and $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$, by recurrence, we know that G_1 has a weight in $(\varrho|_{G_1}, V)$, i.e. there exists a smooth homomorphism $\chi : G_1 \to \mathbb{C}^*$ such that $V_\chi := \{v \in V \mid \varrho(h)v = \chi(h)v\ \forall h \in G_1\} \neq 0$. By Lemma 3.22, this implies that V_χ is G-invariant, i.e. $\varrho(G)V_\chi \subseteq V_\chi$. By Lemma 3.21, this in turn implies that $D_e\varrho(\mathfrak{g})(V_\chi) \subseteq V_\chi$. We already know that $\mathfrak{g} = \mathfrak{g}_1 + \langle Y \rangle$ and that every $v \in V_\chi$ is an eigenvector of $D_e\varrho(X)$ for all $X \in \mathfrak{g}_1$. Now, as $D_e\varrho(Y)(V_\chi) \subseteq V_\chi$, let $v_0 \in V_\chi$ be an eigenvector of $D_e\varrho(Y)$. Then v_0 is an eigenvector of $D_e\varrho(\mathfrak{g})$ and hence $\langle v_0 \rangle \subseteq V$ is $D_e\varrho(\mathfrak{g})$-invariant, thus $\varrho(G)$-invariant by Lemma 3.21. This completes the proof. □

Appendix. Qualitative Baker-Campbell-Hausdorff And Applications

Let G be a Lie group with Lie algebra $\mathfrak{g}$. As announced earlier, we now take a closer look at the local inverse of the exponential map $\exp : \mathfrak{g} \to G$. Namely, if $X, Y \in \mathfrak{g}$ are small enough with respect to so some fixed norm on $\mathfrak{g}$, we would like to determine $\eta(X, Y) \in \mathfrak{g}$ such that $\exp X \exp Y = \exp \eta(X, Y)$. It turns out that η is an analytic function, the first few terms of the power series being

$$\eta(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[Y, [Y, X]] + \frac{1}{12}[X, [X, Y]] + \cdots .$$

The remaining terms are scalar multiples of higher order nested brackets in X and Y. In particular, $\eta(X, Y)$ is contained in the Lie subalgebra of $\mathfrak{g}$ generated by X and Y, which is what we shall prove, using the following two lemmas.

We remark that for any Lie group G, the exponential map can be used to equip G with an analytic atlas (i.e. transition maps are real analytic) and with respect to which it is analytic, hence it makes sense to talk about analytic maps between Lie groups.

Lemma 3.23. Let G be a Lie group, $p \in G$, $U \in \mathcal{U}(p)$ open and let $f : U \to \mathbb{R}$ be an analytic map. Then for all $X \in \mathfrak{g}$ we have

$$f(p \exp tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(X^L\right)^n f(p) \quad \text{for small } t.$$

Here, $(X^L)^n = X^L \circ \cdots \circ X^L$ is the n-fold composition of X^L with itself, viewed as a derivation.

Proof. Using the Taylor expansion of the map $\mathbb{R} \to \mathbb{R}$, $t \mapsto f(p \exp tX)$ at $t = 0$, all one has to show is that

$$\left.\frac{d^n}{dt^n}\right|_{t=0} f(p \exp(tX)) = (X^L)^n f(p)$$

which can be done by induction on n. The induction basis reads

$$\left.\frac{d}{dt}\right|_{t=0} f(p \exp tX) = X^L f(p)$$

which we have used before. $\square$

Lemma 3.24. Let G be a Lie group, $p \in G$, $U \in \mathcal{U}(p)$ open and let $f : U \to \mathbb{R}$ be an analytic map. Then for all small enough $X_1, \ldots, X_n \in \mathfrak{g}$ we have

$$f(p \exp X_1 \cdots \exp X_n) = \sum_{j_1,\ldots,j_n \geq 0} \frac{(X_1^L)^{j_1} \cdots (X_n^L)^{j_n} f(p)}{j_1! \cdots j_n!}.$$

Proof. The case $n = 1$ is exactly Lemma 3.23 for $t = 1$. For $n = 2$, applying Lemma 3.23 in the case $t = 1$ twice yields

$$f(p \exp X_1 \exp X_2) = \sum_{j_2 \geq 0} \frac{(X_2^L f(p \exp X_1))^{j_2}}{j_2!} = \sum_{j_2 \geq 0} \sum_{j_1 \geq 0} \frac{(X_1^L)^{j_1} (X_2^L)^{j_2} f(p)}{j_2! j_1!}.$$

Continuing in this fashion, proves the assertion. $\square$

We now come back to the discussion of $\eta(X, Y)$; we will prove the following theorem which has many interesting consequences.

Theorem 3.25 (Baker-Campbell-Hausdorff). Let G be a Lie group with Lie algebra $\mathfrak{g}$. Pick $B(0, \varepsilon) \subseteq \mathfrak{g}$ and $W \in \mathcal{U}(e)$ such that $\exp : B(0, \varepsilon) \to W$ is an analytic diffeomorphism. Further, let $X, Y \in \mathfrak{g}$ be such that $\exp X \exp Y \in W$. Then $\eta(X, Y) = \log(\exp X \exp Y)$ is a convergent series of vectors of the Lie subalgebra of $\mathfrak{g}$ generated by X and Y.

Note that by Lemma a vector-valued version of 3.24, we have

$$\eta(X, Y) = \log(\exp X \exp Y) = \sum_{n,m \geq 0} \frac{(X^L)^n (Y^L)^m \log(e)}{n! m!}.$$

Therefore, we have to work on the expressions $(X^L)^n (Y^L)^m \log(e)$. To this end, it is natural to work in the space $\operatorname{End} C^\infty(W)$ which contains $\mathcal{L}(G) := \{X^L \mid X \in \mathfrak{g}\}$. The proof of Theorem 3.25 will be immediate from the following two lemmas for which we introduce the following notation: Given $f_1, \ldots, f_p \in \operatorname{End} C^\infty(W)$, we define $S(f_1, \ldots, f_p) := \sum_{\sigma \in S_p} f_{\sigma(1)} \cdots f_{\sigma(p)}$ where S_p is the symmetric group on $\{1, \ldots, p\}$.

Lemma 3.26. Let $\alpha_1, \ldots, \alpha_n \in \mathcal{L}(G)$ and let $L(\alpha_1, \ldots, \alpha_n) \leq \mathcal{L}(G)$ be the Lie subalgebra of $\mathcal{L}(G)$ generated by $\alpha_1, \ldots, \alpha_n$. Then there are elements

$$(f_j^i)_{i,j} \in L(\alpha_1, \ldots, \alpha_n), \quad i \in J,\ 1 \leq j \leq p_i,\ 1 \leq p_i \leq n$$

such that

$$\alpha_1 \cdots \alpha_n = \sum_{i \in J} S(f_1^i, \ldots, f_{p_i}^i).$$

Note that the case $p_i = 1$ includes the possibility of a summand in $L(\alpha_1, \ldots, \alpha_n)$, in particular nested brackets.

Proof. (Idea). Consider the case $n = 2$. Then

$$\alpha_1\alpha_2 = \frac{1}{2}(\alpha_1\alpha_2 + \alpha_2\alpha_1) + \frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1) = S\left(\frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_2}{\sqrt{2}}\right) + \underbrace{\frac{1}{2}[\alpha_1, \alpha_2]}_{\in L(\alpha_1, \ldots, \alpha_n)}.$$

From this, it follows that for all $\sigma \in S_n$ we have $\alpha_1 \cdots \alpha_n = \alpha_{\sigma_1} \cdots \alpha_{\sigma_n} + R_\sigma$ where R_σ is a sum of products of length at most $n-1$ of elements of $L(\alpha_1, \ldots, \alpha_n)$. Do this for a transposition and then write any $\sigma \in S_n$ as a product of transpositions. Repeat. □

The reason why Lemma 3.26 is interesting is the following incredible observation.

Lemma 3.27. Given $g_1, \ldots, g_p \in \mathcal{L}(G)$ we have

$$S(g_1, \ldots, g_p)\log(e) = \begin{cases} g_1 & p = 1 \\ 0 & p > 1 \end{cases}.$$

Proof. Using Lemma 3.23 we have for small $t_i \in \mathbb{R},\ i\{1, \ldots, p\}$:

$$\sum_{i=1}^p t_i g_i = \log\exp\left(\sum_{i=1}^p t_i g_i\right) = \sum_{j=0}^\infty \frac{1}{j!}\left(\sum_{i=1}^p t_i g_i\right)^j \log(e).$$

Comparing coefficients for $j = p$ yields $0 = \frac{1}{p!}S(g_1, \ldots, g_p)$ and hence the assertion. □

Theorem 3.25 now follows from

$$\eta(X, Y) = \sum_{n,m \geq 0} \frac{(X^L)^n (Y^L)^m \log(e)}{n!m!}$$

using Lemma 3.26 and 3.27. In fact, keeping track of the symmetrizations we do in principle get the whole power series. For instance, in the cases

$$(n, m) \in \{(0,0), (1,0), (0,1), (1,1)\}$$

we obtain, using the proof of Lemma 3.26,

$$\eta(X, Y) = 0 + X + Y + \frac{1}{2}[X, Y] + \ldots.$$

3.2. **Lie correspondence.** We now apply Theorem 3.25 to shed some light on the Lie correspondence. For instance, given a Lie group G with Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there should be a "Lie subgroup" corresponding to it. The question is, what the right definition of "Lie subgroup" is. From what we know, "closed subgroup" would be a candidate but there is the following example: Consider the case of the torus $G = S^1 \times S^1$ and an irrational line $\mathbb{R} \leq \mathbb{R}^2 = \mathfrak{g}$. Exponentiating this line yields a proper, dense, and hence non-closed, subgroup; on this real line, the topology induced from the torus topology does not coincide with the Euclidean topology. It turns out, that the right definition of "Lie subgroup" is the following.

Definition 3.28. Let G be a Lie group. A *Lie subgroup* of G is a pair (H, i) consisting of a Lie group H and a smooth, injective, immersive homomorphism $i : H \to G$. Here, immersive means that the differential is everywhere injective.

With this definition we have the following theorem based on Theorem 3.25.

Theorem 3.29. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then there is a Lie subgroup (H, i) of G with $D_e i(\mathrm{Lie}(H)) = \mathfrak{h}$. In fact, $i(H)$ is the subgroup of G generated by $\{\exp X \mid X \in \mathfrak{h}\}$ and $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp tX \in i(H)\ \forall t \in \mathbb{R}\}$.

Proof. (Idea). First, we need to find H. Let us put $H = \langle\{\exp X \mid X \in \mathfrak{h}\}\rangle$. Next, we have to put the right topology on H. This can be done in such a way that the set $\{\exp B_{\mathfrak{h}}(0, \frac{1}{n}) \mid n \in \mathbb{N}\}$ forms a fundamental system of neighbourhoods of $e \in H$. Also, the exponential map $\exp : \mathfrak{h} \to H$ serves to show that H is a manifold. But so far, everything could have been done with $\mathfrak{h}$ being just a vector subspace of $\mathfrak{g}$. The assumption, that it is indeed a Lie subalgebra, comes in when trying to prove that the group multiplication in H is continuous/smooth with respect to the given topology/atlas. For instance, one needs to show that for all $\varepsilon > 0$ there is $\delta > 0$ such that $\exp B_{\mathfrak{h}}(0, \delta) \exp B_{\mathfrak{h}}(0, \delta) \subseteq \exp B_{\mathfrak{h}}(\varepsilon)$. The problem here is not to find a δ for a given ε but to show that the left hand side is still contained in the image of the exponential map, which it is just by Theorem 3.25! □

Overall, we now know that given a Lie group G with Lie algebra $\mathfrak{g}$, to every closed subgroup $H \leq G$ (in which case (H, i), where $i : H \to G$ is the canonical inclusion, is a Lie subgroup in the sense of definition 3.28) corresponds a Lie subalgebra of $\mathfrak{g}$ and that, conversely, to every Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ corresponds a Lie subgroup (H, i) of G.

3.3. **Integration of Lie Algebra Homomorphisms.** Another natural question is: Given Lie groups G, H with Lie algebras $\mathfrak{g}, \mathfrak{h}$, under which circumstances does a Lie algebra homomorphism $\varphi : \mathfrak{h} \to \mathfrak{g}$ integrate to a smooth homomorphism of Lie groups such that the following diagram commutes?

$$\begin{tikzcd} H \arrow[r, "?"] & G \\ \mathfrak{h} \arrow[u, "\exp"] \arrow[r, "\varphi"'] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

This is certainly not always the case, for instance, consider the following:

$$\begin{tikzcd} S^1 \arrow[r, "?"] & \mathbb{R} \\ \mathbb{R} \arrow[u, "\exp"] \arrow[r, "\mathrm{id}"'] & \mathbb{R} \arrow[u, "\exp"'] \end{tikzcd}$$

There is no smooth map from S^1 to $\mathbb{R}$ whose differential is everywhere the identity. For instance, such a map would assume its maximal value at some $\varphi_0 \in S^1$ at which the differential has to be zero.

However, a homomorphism of Lie algebras $\varphi : \mathfrak{h} \to \mathfrak{g}$ can always be integrated by passing to a homomorphism from a covering of H to G as follows. Consider the graph $\mathrm{graph}(\varphi) := \{(X, \varphi(X)) \mid X \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{g}$ of φ which is a Lie subalgebra of the product $\mathfrak{h} \times \mathfrak{g}$. Hence, by Theorem 3.29, there is a Lie subgroup L of $H \times G$ corresponding to it, which we would like to be the graph of a smooth homomorphism from H to G which in general it just is not as we saw in the torus example.

$$\begin{tikzcd}
 & G & \mathfrak{g} \arrow[l, "\exp"'] & \\
H & H \times G \arrow[l, "\mathrm{pr}_H"'] \arrow[u, "\mathrm{pr}_G"] & \mathfrak{h} \times \mathfrak{g} \arrow[l, "\exp"'] \arrow[u, "\mathrm{pr}_{\mathfrak{g}}"'] \arrow[r, "\mathrm{pr}_{\mathfrak{h}}"] & \mathfrak{h} \\
 & L \arrow[u, tail] \arrow[ul, "\mathrm{pr}_H |_L"] & \mathrm{graph}(\varphi) \arrow[l, "\exp"] \arrow[u, tail] &
\end{tikzcd}$$

However, in this situation the map $\mathrm{pr}_H |_L$, which induces an isomorphism of the corresponding Lie algebras, is a covering map and $\mathrm{pr}_G |_L$ is the claimed integrated homomorphism.

3.4. **Lie Group To A Given Lie Algebra.** There still is a third question: Given a Lie algebra $\mathfrak{g}$, is there a Lie group G which has Lie algebra $\mathfrak{g}$) The affirmative answer to this was given by Ado.

Theorem 3.30 (Ado). Any real, finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

Since $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$, the assertion follows from Theorem 3.29. Ado's theorem is proven using the general structure theory of Lie algebras.

Lemma 3.31. Let $\mathfrak{g}$ be a real, finite-dimensional Lie algebra. Then there is a unique maximal solvable ideal in $\mathfrak{g}$.

Proof. Note, that if $\mathfrak{a}_1, \mathfrak{a}_2$ are solvable ideals in $\mathfrak{g}$, then so is $\langle \mathfrak{a}_1, \mathfrak{a}_2 \rangle$. Hence a dimension argument gives the result. □

The unique maximal solvable ideal of a given real, finite-dimensional Lie algebra $\mathfrak{g}$ is called the *(solvable) radical* of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g}) \trianglelefteq \mathfrak{g}$. It is readily checked that $\mathfrak{g} / \mathrm{rad}(\mathfrak{g})$ has zero radical. Lie algebras with vanishing radical are termed *semisimple*. In general, *Levi decomposition* states that any Lie algebra $\mathfrak{g}$ as above is a semidirect product of a semisimple Lie subalgebra $\mathfrak{s}$ and its radical: $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$. The proof of Ado's theorem separately analyzes the solvable radical and the semisimple part. Whereas the former is dealt with using Lie's theorem, the latter can be made tractable using combinatorial objects called *Dynkin diagrams*.

Eventually, semisimple Lie groups occur as isometry groups of Riemannian symmetric spaces, certain quotients of which, such as $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2$ are deeply connected to questions in arithmetic, e.g. to Fermat's last problem.

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