Introduction to Lie Groups

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(HG D5.2 , D 3.2)

Home page:

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This course will consist of three chapters.

I. Topological groups

Prerequisites: basic notions from group theory, point set topology and elementary measure theory.

II. Lie groups: ~~basic differential geometry.~~

Prerequisites: basic differential geometry.

III. Structure theory.

General introduction: the originer of Lie groups com from the problem of solving differential equations. Mention that Lie was motivated by Galois theory: expectations remained unfulfilled. However Lie theory developed in unexpected ways. Applications:

- algebraic topology
- number theory
- geometry.

Chapter I. Topological Groups.

I.1 Topological groups: definitions and examples.

Given a group G we usually denote by $e \in G$ the neutral element, by

$$G \times G \to G$$
$$(x, y) \mapsto x \cdot y$$

the product and by

$$G \to G$$
$$g \mapsto g^{-1}$$

the inverse.

Let $\mathcal{P}(G)$ denote the power set of G.

Def. I.1 A topology $\mathcal{T} \subset \mathcal{P}(G)$ endows G with the structure of topological group if the product $G \times G \to G$, $(x, y) \mapsto x \cdot y$ and the inverse $G \to G$, $g \mapsto g^{-1}$ are continuous maps.

Here $G \times G$ is endowed with the product topology.

Denote for $g \in G$ by $L_g : G \to G$, $x \mapsto g \cdot x$,

called left translation, $R_g : G \to G$, $x \mapsto xg$

right translation and $i : G \to G$, $x \mapsto x^{-1}$

the inverse.

We have

<u>Lemma I.2</u> : If G is a topological group then $\forall g \in G$, L_g, R_g, and i are homeomorphisms.

<u>Proof</u> : (sketch) L_g is continuous being the composition of continuous maps, namely

$$G \longrightarrow G \times G \longrightarrow G$$
$$x \longmapsto (g, x) \longrightarrow g \cdot x .$$

Since $L_g \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = Id_G$, L_g is a homeomorphism. The rest is left as exercise. $\boxed{\xi}$

Notation: given subsets A, B in G, let

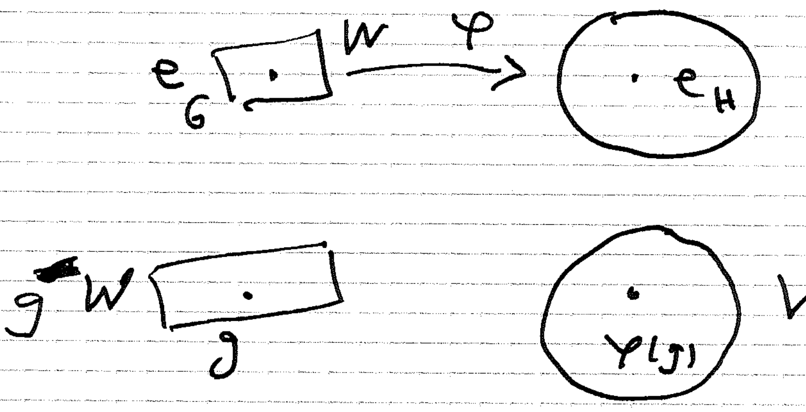
$A \cdot B = \{a \cdot b : a \in A,\ b \in B\}$, in particular

$L_g(B) = g \cdot B$, $R_g(B) = B \cdot g$.

The preceding lemma has the following miraculous consequence:

<u>Lemma I.3</u> Let G, H be topological groups. A homomorphism $\varphi : G \to H$ is continuous iff it is continuous at $e \in G$.

<u>Proof</u>: $\Leftarrow$ Pick $g \in G$ and $V \subset H$ open with $\varphi(g) \in V$:



$\varphi(g)^{-1} V = L_{\varphi(g)^{-1}}(V) \ni e_H$

Then $\varphi(g)^{-1} V \ni e_H$ is open. Pick $W \ni e_G$ open with $\varphi(W) \subset \varphi(g)^{-1} V$. Then $gW = L_g(W) \ni g$ is open and: $\varphi(gW) = \varphi(g)\varphi(W)$ $\subset \varphi(g)\varphi(g)^{-1} V = V$. $\blacksquare$

The following is left as exercise:

Exercises I.4

(1) G top. group, $H < G$ subgroup, then H with induced topology is a topological group.

(2) if G_1, G_2 are top-groups, then $G_1 \times G_2$ with product topology is a topological group.

Now we are going to treat a certain number of examples, some more involved. Among other properties we will examine whether they are compact or locally compact. Locally compact groups turn out to be a very important class since they carry a Haar measure.

~~Examples I.5~~

~~(1) any group G with discrete topology.~~

~~(2) $(\mathbb{R}^n, +)$ where $\mathbb{R}^n$ has the euclidean topology.~~

Recall

Def. I.5 (Munkres p. 182) A top. space X is locally compact if every point admits a compact neighborhood.

And recall

Lemma I.6 (Munkres Thm 29.2) Let X be Hausdorff and locally compact. Then for every $x \in X$, every neighborhood of x contains a compact neighborhood of x.

Examples I.7

(1) Any group with discrete topology ~~is Hausdorff~~.

(2) $(\mathbb{R}^n, +)$, where $\mathbb{R}^n$ has euclidean topology ~~is Hausdorff~~.

(3) $(\mathbb{R}^\times, \cdot)$ and $(\mathbb{C}^\times, \cdot)$.

Let $\mathbb{F}$ be a field; $M_{n,m}(\mathbb{F})$ is the vectorspace of $n \times m$ matrices,

$GL(n, \mathbb{F}) = \{ A \in M_{n,n}(\mathbb{F}) : \det A \neq 0 \}$

and $SL(n,\mathbb{F}) = \{ A \in M_{n,n}(\mathbb{F}) : \det A = 1 \}$.
Then for the matrix product $GL(n,\mathbb{F})$ is a group and $SL(n,\mathbb{F})$ is a normal subgroup of $GL(n,\mathbb{F})$.

If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ then $M_{n,m}(\mathbb{F})$ acquires a topology via the vectorspace isomorphism

$$M_{n,m}(\mathbb{F}) \xrightarrow{\sim} \mathbb{F}^{n \cdot m}.$$

(4) $GL(n,\mathbb{F})$ is a topological group, $\det : GL(n,\mathbb{F}) \to \mathbb{F}^{\times}$ is a continuous homomorphism and $SL(n,\mathbb{F})$ is a closed subgroup.

(5) Let $n = p+q$ and $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the bilinear symmetric form

$$B(x,y) = -\sum_{i=1}^{p} x_i y_i + \sum_{p+1}^{q} x_j y_j$$

and $$J_{p,q} = \begin{pmatrix} -Id_p & 0 \\ 0 & Id_q \end{pmatrix}$$

Then $O(p,q) := \{ g \in GL(n,\mathbb{R}) : B(gx, gy) = B(x,y) \ \forall x, y \in \mathbb{R}^n \}$

$$= \{ g \in GL(n,\mathbb{R}) : {}^t g J_{p,q} g = J_{p,q} \}$$

Then $O(p,q)$ is a closed subgroup of $GL(n,\mathbb{R})$.

Let's discuss (local)-compactness of the examples treated so far.

(1) is l.c. Hausdorff.

(2) $\mathbb{R}^n$ is a l.c. Hausdorff space.

(3) $\mathbb{R}^\times$ is open in $\mathbb{R}$, so is $\mathbb{C}^\times$ in $\mathbb{C}$ as a result both are l.c. H.

(4) $GL(n,\mathbb{F})$ is an open subset of $M_{n,n}(\mathbb{F})$ and $SL(n,\mathbb{F})$ is closed in both $GL(n,\mathbb{F})$ and $M_{n,n}(\mathbb{F})$ thus these groups are l.c.H.

(5) for the same reasons $O(p,q)$ is l.c.H.

And $O(p,q)$ is compact iff $p=0$ or $q=0$.

$\Leftarrow$ $p=0,\ q=n$ $\quad O(n) = \{ g \in GL(n,\mathbb{R}) :$

$${}^t g g = Id \}$$

$$= \{ g \in M_{n,n}(\mathbb{R}) : {}^t g g = Id \}$$

If we compute for a general matrix $X = (x_{ij})$

$$Tr({}^t x x) = \sum_{i,j} x_{ij}^2 .$$

Thus $O(n) \subset \{ X \in M_{n,n}(\mathbb{R}) : \sum_{i,j} x_{ij}^2 = n \}$

and being closed and bounded in $M_{n,n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ is hence compact.

$\Longrightarrow$ exercise.

Example I.8

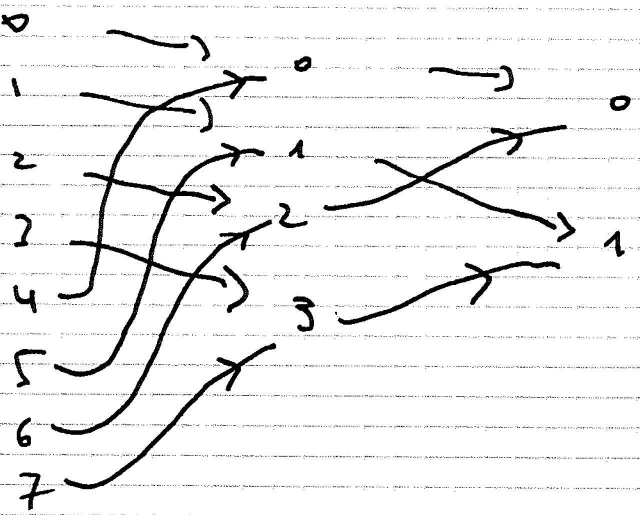
Let p be a prime. For every $n \geq 1$, endow $(\mathbb{Z}/p^n\mathbb{Z}, +)$ with the discrete topology. It is clearly compact as well. For every $n \geq 1$ we have a surjective group homomorphism

$$\varphi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

of reduction mod p^n.

For example:

$$\mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$



Then $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$, the projective limit is roughly the set of sequences $(x_n)_{n \geq 1}$ such that $\varphi_n(x_{n+1}) = x_n \quad \forall n \geq 1$.

Formally, consider the product $\prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$ with product topology, which is a compact topological group. Then

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \Big\{ (x_n)_{n \geq 1} \in \prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z} : \varphi_n(x_{n+1}) = x_n, \ \forall n \geq 1 \Big\}.$$

This is a closed subgroup of $\prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$ and is therefore compact.

Let $\pi_n : \mathbb{Z}_p \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ be the projection on the n'th factor. Clearly $\{0\}$ being open and closed, $\operatorname{Ker}\pi_n < \mathbb{Z}_p$ is an open and closed subgroup. In fact.

$\operatorname{Ker}\pi_n = \{(y_k)_{k\geq 1} \in \mathbb{Z}_p : \quad y_1 = 0, y_2 = 0, \dots; y_n = 0\}$

forms a fundamental system of neighborhoods of $\mathbb{Z}_p$.

The group $\mathbb{Z}$ injects into $\mathbb{Z}_p$ via:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}_p \\ t &\longmapsto (t) \end{aligned}$$

and the image is dense.

One can go a little further by exploiting that $\mathbb{Z}/p^n\mathbb{Z}$ is a ring. The pointwise product on sequences defines then a ring structure on $\mathbb{Z}_p$

for which the product is continuous; the identity is of course $(1) = 1$. In fact one can show that $\mathbb{Z}_p$ is an integral domain. Its field of fractions $\mathbb{Q}_p$ is the field of p-adic numbers, and in fact $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$, is the ring of polynomials in $1/p$ with coefficients in $\mathbb{Z}_p$.

Thus $\mathbb{Q}_p = \bigcup_{n \geq 1} 1/p^n \, \mathbb{Z}_p$ as increasing union and one defines a topology on $\mathbb{Q}_p$ by declaring a set $V \subset \mathbb{Q}_p$ open if $V \cap 1/p^n \mathbb{Z}_p = 1/p^n \underbrace{[p^n V \cap \mathbb{Z}_p]}_{\text{open}}$ $\forall n \geq 1$. This makes $\mathbb{Q}_p$ is a locally compact field for which $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ injects with dense image.

One obtains then in a similar manner that $GL(n, \mathbb{Q}_p)$ and $SL(n, \mathbb{Q}_p)$ are locally compact Hausdorff groups.

This leads to "very large groups"; for instance Homeo(S^1) is not locally compact.

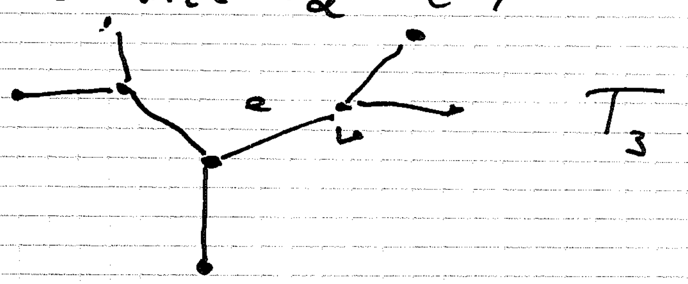
There is a class of examples where the compact open topology leads to locally compact groups. Let (X, d) a metric space. An isometry is a bijective map $f: X \to X$ such that

$$d(f(x), f(y)) = d(x,y) \quad \forall x, y \in X.$$

Let Isom(X) denote the set of all isometries. It is clearly a group for composition and contained in Homeo(X).

<u>Example I.10</u> Assume that $\forall\, 0 \leq r < \infty$ and $x \in X$, $B_{\leq r}(x) = \{y \in X : d(x,y) \leq r\}$ is compact. Then Isom(X) is a locally compact group, as follows from the Arzela-Ascoli theorem.

As an example take ~~the~~ $T_d = (V, E)$ the d-regular tree:



Here V is the set of vertices and E the set of edges. Our metric space is V endowed with the combinatorial distance d. The resulting group of isometries denoted $\mathrm{Aut}\, T_d$ is locally compact. This is a completely "new" class of groups.

I.2. Some (miraculous) facts about topological groups.

In this section we adress some issues concerning connectedness in topological groups. Recall that a topological space X is connected if it is not the disjoint union of two proper open (closed) subsets. A subset of a topological space is connected if it is connected in the induced topology. Then

Facts (1) the closure of a connected set is connected.

(2) the relation $x \sim y$ if $\{x, y\}$ is contained in a connected set is an

equivalence relation on X and its equivalence classes are the connected components of X. Connected components are closed and form a partition of X.

In particular if $C(x)$ denotes the connected component of $x \in X$ and $A \subset X$ is a connected subset with $A \ni x$ then $A \subset C(x)$.

<u>Proposition I.11</u> Let G be a topological group.

(1) the closure $\overline{H}$ of a subgroup $H < G$ is a subgroup.

(2) if a subgroup $H < G$ is open, it is closed as well.

(3) the connected component G^{0} of $e \in G$ is a closed normal subgroup.

(4) if G is connected and $V \ni e$ is a neighborhood of e, then $G = \bigcup_{n \geq 1} (V \cup V^{-1})^{n}$.

(5) if G is connected and $N \triangleleft G$ is a discrete normal subgroup, then $N \subset Z(G)$.

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Notations: $A \subset G$, $A^{-1} = \{a^{-1} : a \in A\}$.

$$A^n := \{a_1 \cdots a_n : a_i \in A\}.$$

If $g \in G$, $\mathrm{Int}(g) : G \to G$, $x \mapsto gxg^{-1}$ is an automorphism of G; it is continuous with continuous inverse $\mathrm{Int}(g^{-1})$.

Proof:

(1) Recall that if $f : X \to Y$ is a continuous map between topological spaces then $\forall A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

Let then for the sake of the proof $m : G \times G \to G$ denote the product map. We have: $m(\bar{H} \times \bar{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)} = \bar{H}$

and $\bar{H}^{-1} = i(\bar{H}) \subset \overline{i(H)} = \bar{H}$.

This shows (1).

(2) Let $\mathcal{R}$ be a set of representatives of the set $G/H = \{gH : g \in G\}$ of $\binom{\text{right}}{\text{left}}$ cosets, with $\mathcal{R} \ni e$.

Then $G = \bigsqcup_{x \in R} xH = H \sqcup \bigsqcup_{\substack{x \in R \\ x \neq e}} xH$.

Then xH is open, hence $\bigsqcup_{\substack{x \in R \\ x \neq e}} xH$ as well

which implies that H is closed.

(3) Since connected components are closed, G° is closed.

Since $G^\circ \times G^\circ \subset G \times G$ is connected,

$m(G^\circ \times G^\circ) \ni e$ is connected hence $m(G^\circ \times G^\circ) \subset G^\circ$. ~~that~~ The same argument shows that $i(G^\circ) \subset G^\circ$. Thus G° is a subgroup.

Finally: $\mathrm{Int}(g)(G^\circ) \ni e$ is connected hence $\mathrm{Int}(g)(G^\circ) \subset G^\circ$, for all $g \in G$ which shows that G° is a normal subgroup.

(4) Define $H := \bigcup_{n \geq 1} (V \cup V^{-1})^n$.

Since $(V \cup V^{-1})^n (V \cup V^{-1})^m \subset (V \cup V^{-1})^{n+m}$

and $[(V \cup V^{-1})^n]^{-1} = (V \cup V^{-1})^n$

we conclude that H is a subgroup.

Since $H \supset V$, H is a neighborhood of e. But then $H = hH \ni h \cdot e$ is a neighborhood of h, $\forall h \in H$. Hence H is an open subgroup hence closed. Thus $G \setminus H$ is open and closed, and since G is connected and $H \neq \emptyset$ this implies $H = G$.

(5) Pick $n \in N$ and consider the map

$$\varphi_n : G \longrightarrow N$$
$$g \mapsto gng^{-1}$$

which is continuous. Since N is discrete, $\{n\} \subset N$ is open and hence

$$\varphi_n^{-1}(n) = \{ g \in G : gng^{-1} = n \} \text{ is open.}$$

But it is also a subgroup. Since G is connected this implies $\varphi_n^{-1}(n) = G$. $\blacksquare$

<u>Example I.12</u> This example requires some knowledge of covering theory. Let G be a topological group that is connected and locally path connected, and $(\tilde{G}, \tilde{e})$ the universal covering of (G, e) and $p : \tilde{G} \to G$ the covering

projection. The maps $m : G \times G \to G$

$i : G \to G$

lift to $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ $(\tilde{e}, \tilde{e}) \mapsto \tilde{e}$

$\tilde{i} : \tilde{G} \to \tilde{G}$ $\tilde{e} \mapsto \tilde{e}$

uniquely and one shows that they endow $\tilde{G}$ with the structure of topological group for which $p : \tilde{G} \to G$ is a continuous isomorphism. The fiber $p^{-1}(e) = \ker p$ is then identified with $\pi_1(G)$; it is a discrete normal subgroup hence contained in $Z(\tilde{G})$. In particular $\pi_1(G)$ is abelian.

<u>Example I.13</u> Since $G^0 \triangleleft G$, the set $\pi_0(G)$ of connected comp. is identified with G/G^0 and acquires a group structure which however has no reason to be abelian.

We close this section by discussing connectedness as it relates to our example list. First a remark: if G is a topological group and

$G^\circ = \{e\}$ then clearly for every $g \in G$, $C(g) = gG^\circ = \{g\}$; thus the connected components are reduced to points and we say that G is totally disconnected.

(1) a discrete group is t-d.

(2) F finite and $F^{\mathbb{N}}$ are totally disc.

(3) $\mathbb{R}^n$ is connected.

(4) $G = \mathbb{R}^\times$, $G^\circ = \mathbb{R}_{>0}$ and $\mathbb{R}^\times / \mathbb{R}_{>0} \simeq \mathbb{Z}/2\mathbb{Z}$.
$\mathbb{C}^\times$ is connected.

(5) $\det : GL(n, \mathbb{R}) \to$ ~~[illegible]~~ $\mathbb{R}^\times$ and
$GL(n, \mathbb{R})^\circ = \{A \in GL(n, \mathbb{R}) : \det A > 0\}$.
$SL(n, \mathbb{R})$ is connected.
$GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are connected.

(6) $O(n)^\circ = SO(n)$ and for $1 \leq p, q \leq n$
$O(p,q) / O(p,q)^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

(7) $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are to.d. and so are $GL(n, \mathbb{Q}_p)$ and $SL(n, \mathbb{Q}_p)$.

(8) $\mathrm{Homeo}(S^1)$ has 2 connected components and $\mathrm{Homeo}(S^1)^0 = \{\varphi : S^1 \to S^1 : \varphi$ respects the cyclic order$\}$.

If $\mathbb{T}^2 = S^1 \times S^1$ then

$$\mathrm{Homeo}(\mathbb{T}^2) / \mathrm{Homeo}(\mathbb{T}^2)^0 \cong GL(2, \mathbb{Z}).$$

I. 3. Haar measure.

Let X be l.c.H., $C_{oo}(X) = \{ f : X \to \mathbb{C} :$ f is continuous with compact support $\}$ and assume that a group G acts on X:

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g.x$$

such that $\forall g \in G$, $x \mapsto g.x$ is a homeomorphism. Then every $g \in G$ gives rise via:

$$\lambda(g) f(x) := f(g^{-1}x), \quad f \in C_{oo}(X), x \in X$$

to an element of $GL(C_{oo}(X))$ and moreover

$$\lambda : G \to GL(C_{oo}(X))$$

is a homomorphism. If $C_{oo}(X)'$ denotes the vector space dual we obtain by:

$$\lambda^*(g)(m)(f) = m(\lambda(g)^{-1} f), \quad f \in C_{oo}(X)$$

an element in $GL(C(X)')$ and: λ^* is again a homomorphism.

Recall that $\Lambda \in C_{00}(X)'$ is called <u>positive linear functional</u> if whenever $f \in C_{00}(X)$, with $f(X) \subset [0, \infty)$ we have $\Lambda(f) \geq 0$.

Recall the following fundamental

<u>Theorem</u> (Riesz) Let $\Lambda \in C_{00}(X)'$ be a positive linear functional. Then there is a regular Borel measure μ on X such that

$$\Lambda(f) = \int_X f(x)\, d\mu(x) \qquad \forall f \in C_{00}(X).$$

For basic Lebesgue integral and the Riesz repr. theorem we refer to W. Rudin "Real and Complex Analysis". Chap. 1+2.

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<u>Exercise</u> : Let Λ, μ be as in the above thm. Then $\lambda(g)\Lambda$ corresponds to the measure $g_*\mu$ defined by $g_*\mu(E) = \mu(g^{-1}E)$, for every Borel set E.

Specialize now to $X = G$ in which case the action $G \times G \longrightarrow G$ is just multiplication.

<u>Def. I.14</u> A left Haar measure on G is a positive linear functional $m \in C_{00}(G)'$; $m \neq 0$, such that

$$\lambda(g)^* m = m \quad \forall g \in G.$$

If μ is the corresponding Borel measure we will also refer to μ as a left Haar measure on G.

We have then

<u>Thm I.15 (Haar)</u> On every l.c. H. group there is a left Haar measure. It is unique up to positive scalar multiples.

The proof can be found for instance in A. Weil "L'intégration dans les groupes topologiques et ses applications".

Let us give examples where the existence is obvious.

Example I.16

(1) $(\mathbb{R}^n, +)$ the Lebesgue measure,

(2) $(\mathbb{R}_{>0}, \cdot)$: for $f \in C_{00}(\mathbb{R}_{>0})$

$$m(f) = \int_0^\infty \frac{f(x)}{x}\, dx$$

is a left Haar measure.

(3) G discrete; $\mu(E) = \mathrm{Card}(E)$ is a left Haar measure.

Later in this section we will show the uniqueness statement; it has important structural consequences. Observe that even in the above examples (1), (2) uniqueness would require some work.
In this context it will be also useful to define:
$$\rho(y)f(x) = f(xy), \quad f \in C_{00}(X), \; y \in G, \; x \in G$$

and $\rho(y)^*(m)(f) = m(\rho(y)^{-1} f)$

We have analogously the concept of right Haar measure.

This interplay between left and right translation of G on itself will turn out to be very useful.

~~Lemma I.17~~.

Notation: $f \in C_{00}(X)$, $\check{f}(x) = f(x^{-1})$.

Lemma I.17 Let m be a left Haar measure on G. Then: $\ell(f) := m(\check{f})$ is a right Haar measure.

Proof: We compute: $\ell(\rho(g)f) = m((\rho(g)f)^\vee)$

But $(\rho(g)f)^\vee(x) = f(x^{-1}g) = f((g^{-1}x)^{-1}) = \check{f}(g^{-1}x)$

$= \lambda(g)\check{f}$

Which implies $\ell(\rho(g)f) = m(\lambda(g)\check{f}) = m(\check{f})$

$= \ell(f)$. $\blacksquare$

Let X be locally compact and μ a ~~regular~~ positive regular Borel measure.

Recall that the support $\operatorname{supp}(\mu) \subset X$ is the closed subset defined by

$$\operatorname{supp}(\mu) = \{x \in X : \text{for every open } U \ni x, \ \mu(U) > 0\}.$$

In order to prove uniqueness of the left Haar measure we will need

<u>Lemma I.18.</u> Let m be a left Haar measure on G and let μ be the corresponding Borel measure.

(1) $\operatorname{supp}\mu = G$. In particular $\forall V \subset G$ open with $V \neq \emptyset$, $\mu(V) > 0$.

(2) Let $h \in C(G)$ be continuous with

$$\int_G h(x)\varphi(x)\,d\mu(x) = 0 \quad \forall \varphi \in C_{00}(G).$$

Then $h = 0$.

<u>Proof:</u> (1) First, $\operatorname{supp}\mu \neq \emptyset$. Then observe

that if $x \in \operatorname{supp}\mu$ and $V \ni gx$ is open then $g^{-1}V \ni x$ is open, hence ~~[illegible]~~ $\mu(V) = \mu(g^{-1}V) > 0$ which implies $gx \in \operatorname{supp}\mu$ hence $\operatorname{supp}\mu = G$.

(2) The first property has the following important consequence: if $f : G \to \mathbb{C}$ is continuous and $f(g) = 0$ for μ almost every $g \in G$, then $f \equiv 0$.

Indeed: $\{g \in G : f(g) \neq 0\}$ is open and hence has measure zero iff it is empty.

Let now $\varphi = \bar{h} \cdot \psi$, $\psi \in C_{00}(G)$, $\psi \geq 0$.

Then $\int_G |h(g)|^2 \psi(g) \, d\mu(g) = 0$ and by Lebesgue's theorem $|h|^2 \cdot \psi = 0$ almost everywhere hence $|h|^2 \cdot \psi \equiv 0$. Since this holds $\forall \psi \in C_{00}(G)$, with $\psi \geq 0$, we conclude that $h \equiv 0$. $\boxed{\Sigma}$

Proof of Uniqueness part in Thm I.15.

This proof assumes that G is countable union of compactes.
Let m, m' be left Haar measures and $n(\varphi) = m'(\check{\varphi})$ so that n is a right Haar measure.
Let μ resp. ν be the Borel measures corresponding to m resp. n. For $f, \varphi \in C_{00}(G)$ we compute using Fubini and right invariance of n:

$$m(f)n(\varphi) = \int_G f(t)\,d\mu(t) \int_G \varphi(y)\,d\nu(y)$$

$$= \int_G f(t) \underbrace{\int_G \varphi(y)\,d\nu(y)}_{\int_G \varphi(yt)\,d\nu(y)} d\mu(t)$$

$$= \int_G \underbrace{\int_G f(t)\varphi(yt)\,d\mu(t)}_{\int_G f(y^{-1}t)\varphi(t)\,d\mu(t)} d\nu(y)$$

$$= \int_G \left\{ \int_G f(y^{-1}t)\,d\nu(y) \right\} \varphi(t)\,d\mu(t).$$

If $m(f) \neq 0$, let $m_f(t) := \frac{1}{m(f)} \int_G f(y^{-1}t)\, d\nu(y)$.

Then we have:

$$n(\varphi) = \int_G m_f(t)\, \varphi(t)\, d\mu(t)$$

Thus for every $f_1, f_2 \in C_{00}(G)$ with $m(f_1) \neq 0$, $m(f_2) \neq 0$ ~~we have~~ and every $\varphi \in C_{00}(G)$:

$$\int_G \left(m_{f_1}(t) - m_{f_2}(t)\right) \varphi(t)\, d\mu(t) = 0 .$$

Hence $m_{f_1}(t) = m_{f_2}(t)$ $\forall t \in G$ by continuity of $m_f(t)$ (exercise).

Thus $c := m_f(e)$ is independent of the choice of f, as long as $m(f) \neq 0$ which implies

$$m(f) \cdot c = \int_G f(y^{-1})\, d\nu(y) = n(\check{f})$$
$$= m'(f). \quad \blacksquare$$

~~An analogous argument shows that $m(f) = 0$~~ ~~if $m(f) = 0$.~~

If on the other hand $m(f) = 0$, then the computation on p. 31 gives:

$$0 = \int_G \left\{ \int_G f(y^{-1}t)\, d\nu(y) \right\} \varphi(t)\, d\mu(t) \quad \forall \varphi \in C_{00}(G)$$

hence $\int_G f(y^{-1}t)\, d\nu(y) = 0 \quad \forall t \in G$

in particular $m'(f) = 0$. ∎

Let $Aut(G) = \{ \alpha : G \to G : \alpha$ is a group automorphism and α is a homeomorphism$\}$.

Then given a left Haar measure m, and $\alpha \in Aut(G)$,

$$m'(f) = \cancel{\int f\circ\alpha(x)\,d\mu(x)}\ m(f\circ\alpha)$$

obviously defines a left Haar measure by virtue of the identity:

$$(\lambda(g)f)\circ\alpha = \lambda(\alpha^{-1}(g))(f\circ\alpha)$$

Thus there is $\mathrm{mod}_G(\alpha) \in \mathbb{R}_{>0}$ such that

$$m'(f) = \mathrm{mod}_G(\alpha)\, m(f) = m(f\circ\alpha)$$

and $\mathrm{mod}_G(\alpha)$ is clearly independent of the choice of m. The function

$$\mathrm{mod}_G : Aut(G) \longrightarrow \mathbb{R}_{>0} \qquad \Longrightarrow$$

is called the modular function and it is a homomorphism. In particular we can evaluate mod_G on $Int(g)$, and get

The modular function $\Delta_G : G \longrightarrow \mathbb{R}_{>0}$,

$$\Delta_G(g) = \operatorname{mod}_G \operatorname{Int}(g).$$

Explicitely we have for a left Haar measure μ:

$$\int_G f(xg^{-1})\,d\mu(x) = \int_G f(gxg^{-1})\,d\mu(x) = \Delta_G(g)\int_G f(x)\,d\mu(x).$$

The following is left in exercise:

<u>Prop. I.19</u> Let G be a l.c. H. group. Then

(1) $\Delta_G : G \longrightarrow \mathbb{R}^\times_{>0}$ is a continuous homomorphism.

(2) $$\int_G f(x^{-1})\,\Delta_G(x^{-1})\,d\mu(x) = \int_G f(x)\,d\mu(x) \quad \forall f \in C_0(G).$$

<u>Def. I.20</u> G is unimodular if $\Delta_G = 1$, equivalently every left Haar measure is also a right Haar measure.

Examples I.21.

(1) a l.c. abelian group is unimodular.

(2) a discrete group is unimodular.

(3) a compact group is unimodular: indeed if K is compact, $\Delta_K(K)$ is a compact subgroup of $\mathbb{R}_{>0}$ hence $= \{1\}$.

(4) $G = GL(n, \mathbb{R}) \subset M_{n,n}(\mathbb{R})$: ~~consider~~

$$m(f) := \int_G f(X) \, |\det X|^{-n} \, d\lambda(X)$$

$f \in C_{00}(GL(n,\mathbb{R}))$, $d\lambda = \prod_{i,j} dx_{ij}$

defines a left and right Haar measure.

Observe: – since G is open in $M_{n,n}(\mathbb{R})$ this functional is not identically zero.

– the function $G \longrightarrow \mathbb{R}_{>0}$

$X \longmapsto |\det X|$

is bounded on every compact subset of G, hence $m(f)$ is well defined $\forall f \in C_{00}(G)$.

Verification: define $T_g : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$

$$X \longmapsto gX$$

for $g \in GL(n,\mathbb{R})$. Then: $\det T_g = (\det g)^n$.

Next:

$$m(\lambda(g)f) = \int_G f(\cancel{g}\, g^{-1}X)\, |\det X|^{-n}\, d\lambda(X)$$

$$= \underbrace{\int_G f(g^{-1}X)\, |\det g^{-1}X|^{-n}\, d\lambda(X) \cdot |\det g|^{-n}}$$

$$= \int_G f(Y)\, |\det Y|^{-n}\, d\lambda(Y) \cdot \underbrace{|\det T_g|}_{|\det g|^n}\, |\det g|^{-n}.$$

$= m(f)$.

An analogous computation gives right invariance.

Here is a non-unimodular example:

(5) $$P = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x > 0,\ y \in \mathbb{R} \right\}$$

$$m(f) = \int_{\mathbb{R}} \int_0^\infty f\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\right) d\lambda(x)\, d\lambda(y)$$

is a left Haar measure. One computes:

$$\Delta_{\Phi}\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = a^{-2}.$$

We finish this subsection with

<u>Prop. I.22</u>. G l.c. has finite Haar measure iff it is compact.

<u>Proof:</u> $\Longleftarrow$ Regularity.

$\Longrightarrow$ Assume $\mu(G) = 1$. Pick $K \subset G$ compact with $\mu(K) > 1/2$. Then $\forall x \in G$: $xK \cap K \neq \phi$ hence $G = KK^{-1}$. Since KK^{-1} is the image of $K \times K$ ∎

under the continuous map $G \times G \to G$
$(x,y) \mapsto xy^{-1}$

we conclude that G is compact. $\boxed{\Xi}$

I.4. Homogeneous Spaces.

Let G be a ~~topological~~ group and $H < G$ a subgroup. Recall that the set G/H is the set of equivalence classes for the equivalence relation $x \sim y$ if $x^{-1}y \in H$. Thus $G/H = \{gH : g \in G\}$ and we denote by $p: G \to G/H$, $g \mapsto gH$ the canonical projection. The set G/H is the set of right cosets and we have a G-action:

$$G \times G/H \to G/H$$
$$(g, xH) \mapsto gxH.$$

If now G is a topological group, we endow G/H with the quotient topology, that is:

$$U \subset G/H \text{ is open} \iff p^{-1}(U) \subset G \text{ is open.}$$

We have then:

Prop. I.23: Let G be a top. group and $H < G$.

(1) The map $p: G \to G/H$ is open, that is, the image of an open set is open.

(2) The action $G \times G/H \to G/H$ is continuous.

$$(g, xH) \mapsto gxH$$

(3) G/H is Hausdorff iff $H < G$ is closed.

(4) If G is l.c. then so is G/H.

(5) If G is l.c. ~~and~~ and H closed then for every compact $C \subset G/H$ there is a compact set $K \subset G$ with $p(K) = C$.

Proof: (1) Let $V \subset G$ be open. Then:

$$p^{-1}(p(V)) = V \cdot H = \bigcup_{h \in H} Vh \text{ is open.}$$

(2) Let $U \subset G/H$ open. Then:

$\{(g, xH) : gxH \in U\}$ is the image

under $G \times G \longrightarrow G \times G/_H$

$$(g, x) \longmapsto (g, xH)$$

of $\{(g, x) \in G \times G : gx \in p^{-1}(U)\}$

and is hence open.

(3) $G/_H$ Hausdorff $\Rightarrow$ H closed : clear.

Conversely, assume H closed, and let $xH \neq yH$. Then $yHx^{-1} \not\ni e$ and hence there is $V \ni e$ open with ~~[illegible]~~

~~Then $VyH \cap VxH = \phi$~~

$yHx^{-1} \cap W^{-1}W = \phi$. Hence $WyH \cap Wx = \phi$

and hence $WyH \cap WxH = \phi$.

(4) Let $xH \in G/_H$ and $U \ni xH$ open.

Then $p^{-1}(U) \ni x$ is open and hence there is $V \ni x$ open with $x \in V \subset \bar{V} \subset p^{-1}(U)$ and $\bar{V}$ compact. But then $p(\bar{V}) \supset p(V) \ni xH$ is a compact neighborhood of xH.

Assume G l.c.

(5) Let $C \subset G/H$ be compact; let $\{V_\alpha : \alpha \in A\}$ be the family of all open sets with compact closure. As G is l.c. $\bigcup V_\alpha = G$ hence $\bigcup_\alpha p(V_\alpha) \supset C$ and hence there is $F \subset A$ finite with $\bigcup_{\alpha \in F} p(V_\alpha) \supset C$. But now: $p^{-1}(C)$ is closed, $\overline{V}_\alpha \cap p^{-1}(C)$ is compact and so is $K := \bigcup_{\alpha \in F} (\overline{V}_\alpha \cap p^{-1}(C))$ and $p(K) = C$. $\boxed{S}$

4.10.18

Assume now $G \times X \to X$ is a transitive action of a top. group G on a top. space X. Then the orbit map $g \mapsto gx_0$ induces a continuous bijection:

$$G/_{\mathrm{Stab}(x_0)} \longrightarrow X.$$

The question when this map is an actual homeomorphism is tricky. Here is a very general result:

<u>Thm.</u> I.24 Assume G is l.c. H. and separable, and X is loc. compact. Then

$$G/_{\mathrm{Stab}(x)} \longrightarrow X$$

is a homeomorphism.

This is actually an application of the Baire cathegory theorem. (left in exercise).

<u>Examples I.25</u>

(1) $G = SO(n+1, \mathbb{R}) = SO(n+1)$ acting on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ the unit sphere. This group acts transitively on $\mathbb{S}^n$, and $\mathrm{Stab}(e_{n+1}) := H$

$$= \left\{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} : A \in SO(n, \mathbb{R}) \right\}.$$

Then $G/_H$ is homeomorphic to S^n. This can be used to prove by induction that $SO(n, \mathbb{R})$ is connected.

Indeed: if G is a top. group and if H and $G/_H$ are connected, then G is connected.

Thus this is an example where the a priori knowledge of the topology on S^n can be put to use via the homeo. $G/_H \simeq S^n$.

(2) Let $1 \leq p \leq n$ and consider the set

$$F.O(p,n) = \left\{ (f_1, \ldots, f_p) \in (\mathbb{R}^n)^p : f_1, \ldots, f_p \text{ is orthonormal} \right\}.$$

Then $O(n)$ acts transitively on $FO(p,n)$ and $Stab(e_1, \ldots, e_p) = \left\{ \begin{pmatrix} Id_p & \\ & A \end{pmatrix} : A \in O(n-p) \right\}$.

Thus via $O(n)/_{Stab(\ \ )} \longrightarrow FO(p,n)$

one can endow $FO(p,n)$ with a natural topology.

(3) Let $\mathbb{H} = \{ z \in \mathbb{C} : y > 0 \}$ be the upper half plane. Then $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ via:

$$SL(2,\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto \frac{az+b}{cz+d} .$$

by fractional linear / Möbius transformations.

This action is transitive:

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} i = x + iy .$$

$$\operatorname{Stab}(i) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ai + b = -c + di \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = SO(2) .$$

Thus $SL(2,\mathbb{R}) / SO(2) \xrightarrow{\sim} \mathbb{H}$ is a homeo

and : if $A = \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y > 0 \right\}$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

Then $\forall g \in SL(2,\mathbb{R})$: $\exists!\, n \in N,\ a \in A$

$n a i = g.i$, hence $\exists!\, k \in SO(2)$:

$n a k = g$.

This implies $N \times A \times K \longrightarrow SL(2,\mathbb{R})$

$(n, a, k) \longmapsto nak$

is a homeomorphism.

Related to the preceding:

(4) Let $X = Sym_1^+(n) = \{ A \in M_{n\times n}(\mathbb{R}) :$

${}^tA = A$, $\det A = 1$ and $A \gg 0 \}$.

Action: ~~The group~~ $SL(n,\mathbb{R}) \times X \longrightarrow X$

$(g, A) \longmapsto g A {}^tg$.

The theory of positive definite q.f. on $\mathbb{R}^n$ shows that this action is transitive.

As a result: $SL(n,\mathbb{R})/SO(n) \xrightarrow{\sim} Sym_1^+(n)$.

This falls into the class of Riemannian mnfds called symmetric spaces.

(5) Let $\Lambda < \mathbb{R}^n$ be a discrete subgroup such that ~~$\mathbb{R}^n$~~ $\mathbb{R}^n/\Lambda$ is compact. Then (exercise) there exists $\{f_1, \dots, f_n\} \subset \Lambda$ that is a

- basis of $\mathbb{R}^n$
- $\Lambda = \mathbb{Z} f_1 + \dots + \mathbb{Z} f_n$.

If $g_1, \dots, g_n$ is another such basis, then ~~there is $A \in GL(n,\mathbb{Z})$: $g_i = A f_i$~~

$$g_i = \sum_{j=1}^{n} a_{ji} f_j$$

and $A = (a_{ij}) \in GL(n, \mathbb{Z})$. As a result: $|\det(f_1, \dots, f_n)|$ is independent of the choice of such basis, and denoted $\mathrm{Vol}(\mathbb{R}^n/\Lambda)$. Let

$$\mathcal{R}_1 = \{ \Lambda < \mathbb{R}^n : \Lambda \text{ discrete}, \ \mathbb{R}^n/\Lambda \text{ is compact}, \ \mathrm{Vol}(\mathbb{R}^n/\Lambda) = 1 \}.$$

Then we have an action

$$SL(n,\mathbb{R}) \times \mathcal{R}_1 \longrightarrow \mathcal{R}_1$$

$$(g, \Lambda) \mapsto g(\Lambda).$$

It follows from the above that this action is transitive and

$$Stab(\mathbb{Z}^n) = SL(n,\mathbb{Z}).$$

This induces a bijection:

$$SL(n,\mathbb{R})/_{SL(n,\mathbb{Z})} \longrightarrow \mathcal{R}_1.$$

Thus $\mathcal{R}_1$ becomes a l.c.H space on which $SL(n,\mathbb{R})$ acts continuously.

In many aspects of number theory, this space plays a fundamental role.

For instance, endowed with this topology one has the celebrated:

<u>Mahler Compactness Criterion</u>: a subset $S \subset \mathcal{R}_1$ has compact closure if and only if there is $\varepsilon > 0$

such that $\forall \Lambda \in S : \Lambda \cap B(0,\varepsilon) = \{0\}$.
That is the set of lattices in S is uniformly discrete.

The question to which we turn now is: given G l.c.H and $H < G$ closed, under which conditions does there exist a G-invariant positive linear functional on G/H?

Here are two illuminating examples:

Examples I.26

(1) For the $SL(2,\mathbb{R})$-action on $\mathbb{R}^2 \setminus \{0\}$, the Lebesgue measure is inv.

(2) There is no $SL(2,\mathbb{R})$-invariant positive regular Borel measure on $\mathbb{P}^1(\mathbb{R})$.

In the first case $\mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix} = \left\{\begin{pmatrix}1 & t\\0 & 1\end{pmatrix} : t \in \mathbb{R}\right\}$
$= N$

In the second case: denoting $\overline{\begin{pmatrix}1\\0\end{pmatrix}} \in \mathbb{R}^2 \setminus \mathbb{R}$,

$\mathrm{Stab}\left(\overline{\begin{pmatrix}1\\0\end{pmatrix}}\right) = \left\{\begin{pmatrix}a & b\\0 & a^{-1}\end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R}\right\}$

The following Thm. answers completely the question:

<u>Theorem I.27</u> (A. Weil) Let G be l.c.H with left Haar measure μ_G, $H < G$ a closed subgroup with l.H.m. μ_H. Then there is a G-invariant positive regular Borel measure on G/H if and only if $\Delta_G\big|_H = \Delta_H$.

In this ~~case one can choose~~ this measure $\mu_{G/H}$

is unique up to scaling and can be chosen such that

$$\int_G f(g)\, d\mu_G(g) = \int_{G/H} d\mu_{G/H}(\dot{x}) \int_H f(xh)\, d\mu_H(h).$$

$\forall f \in C_{00}(G/H)$.

~~Proof.~~ ~~The most important tool~~

~~Necessity.~~

We start with a preliminary discussion that applies to any closed subgroup $H < G$.

For $f \in C_{00}(G)$, $x \in G$, the function

$$H \longrightarrow \mathbb{C}$$
$$h \longmapsto f(xh)$$

is obviously continuous with compact support so that its μ_H-integral makes sense.

Define: $T_H f(x) := \int_H f(xh)\, d\mu_H(h)$.

Then $T_H f$ defines a function on G/H.
We have

<u>Lemma I.28</u>: For ~~[illegible]~~ every $f \in C_{00}(G)$, $T_H f \in C_{00}(G/H)$ and the resulting linear map $T_H : C_{00}(G) \to C_{00}(G/H)$ is surjective. In addition, if $F \geq 0$ we may choose $f \in C_{00}(G)$, $f \geq 0$ with $T_H f = F$.

Assuming this:

<u>Proof of Thm I.27</u>: 9.10.18

<u>Necessity</u>: If μ ~~[illegible]~~ is a G-invariant regular Borel measure on G/H, define

$$I(f) := \int_{G/H} d\mu(x) \int_H f(xh)\, d\mu_H(h). \qquad (1)$$

Then: I is a left Haar measure, hence

$$I(f) = c \cdot \int_G f(y)\, d\mu_G(y). \qquad (2)$$

Apply this to $\rho(h')f$, ~~...~~ $h' \in H$.

We obtain from

$$(1): \quad I(\rho(h')f) = \int_{G/H} d\mu(x) \underbrace{\int_H f(xhh')\,d\mu_H(h)}_{\Delta_H(h')^{-1} \int_H f(xh)\,d\mu_H(h)}$$

$$= \Delta_H(h')^{-1} I(f)$$

And from 2: $I(\rho(h')f) = \Delta_G(h')^{-1} I(f)$

which shows necessity.

<u>Sufficiency</u>: We are going to define a positive linear functional on $C_{00}(G/H)$.

~~Pick~~

Given $F \in C_{00}(G/H)$, chose $f \in C_{00}(G)$ with

$$T_H f = F$$

and define $J(F) := \int_G f(g)\,d\mu_G(g)$.

We have to show that this is well defined.

To this end we show that if $T_H f = 0$ then $\int_G f(g)\, d\mu_G(g) = 0$. This will imply well-definedness of J.

We have for all $\varphi \in C_{00}(G)$:

$$\int_G \varphi(x) \int_H f(xh)\, d\mu_H(h)\, d\mu_G(x)$$

$$= \int_H \cancel{d\mu_H(h)} \underbrace{\int_G \varphi(x) f(xh)\, d\mu_G(x)}_{\Delta_G(h^{-1}) \int_G \varphi(xh^{-1}) f(x)\, d\mu_G(x)}\, d\mu_H(h)$$

$$= \int_G f(x) \underbrace{\int_H \varphi(xh^{-1}) \Delta_G(h^{-1})\, d\mu_H(h)}_{\int_H \varphi(xh^{-1}) \Delta_H(h^{-1})\, d\mu_H(h)}\, d\mu_G(x)$$

(Prop. I. 13 (2))

$$\| $$

$$\int_H \varphi(xh)\, d\mu_H(h).$$

$$= \int_G f(x) \underbrace{\int_H \varphi(xh)\, d\mu_H(h)}_{T_H\varphi(x).} \, d\mu_G(x).$$

Choose now $\varphi \in C_{00}(G)$ such that $T_H\varphi = 1 \;\; \forall x \in \operatorname{supp}(f)$ (possible by lemma I.28)

implies $\int_G f(x)\, d\mu_G(x) = 0.$

Thus: $J : C_{00}(G) \to \mathbb{C}$

$$F \mapsto \int_G f(x)\, d\mu_G(x)$$

where $T_H f = F$ defines a positive linear functional. In addition: if $T_H f = F$ then $T_H(\lambda(g) f) = \lambda(g) F$ and

$$J(\lambda(g) F) = \int \lambda(g) f(x)\, d\mu_G(x)$$
$$= \int f(x)\, d\mu_G(x) = J(F).$$

∎

Proof of Lemma I.28 (idea)

Let $F \in C_{00}(G/H)$, $p : G \to G/H$ the projection and $K \subset G$ compact with $p(K) = \operatorname{supp} F$. Let $\eta \in C_{00}(G)$ with

$$\chi_K \leq \eta .$$

Then $T_H \eta \in C_{00}(G/H)$, $T_H \eta \geq 0$ and for any $x \in K$:

$$\int_H \eta(xh)\, d\mu_H(h) > 0.$$

Define
$$f(g) = \begin{cases} \dfrac{F \circ p(g)\, \eta(g)}{T_H \eta(g)}, & T_H \eta(g) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then one verifies $f \in C_{00}(G)$ and $T_H f = F$.

$\boxtimes$

Corollary I.29. If $H_1 < H_2 < G$ are closed subgroups ~~the~~ and H_1, H_2, G are unimodular, ~~then there is a choice~~ then there is a choice of invariant measures such that

$$\int_{G/H_1} f(x)\, d\mu_{G/H_1}(x) = \int_{G/H_2} \int_{H_2/H_1} f(yz)\, d\mu_{H_2/H_1}(z)\, d\mu_{G/H_2}(y).$$

This formula is exploited in A. Weil "Sur quelques résultats de Siegel", Summa Brasiliensis Mathematicae 1, 1946, 21-33. He shows among other things, that the formula is valid for all $f : G/H_1 \to [0,\infty)$ continuous. Alternatively: when G is second countable the formula is valid whenever one of the following two is satisfied:

(1) $f : G \to [0,\infty)$ is μ_G-measurable.

(2) $f \in L^1(G/H_1, \mu_{G/H_1})$.

Example I.30 (Minkowski's thm.)

The following is in the context of Thm I.25 (b).

Let $\Lambda < \mathbb{R}^n$ be discrete with $\mathbb{R}^n/\Lambda$ compact; such a subgroup is called a lattice.

Let dx be the Lebesgue measure so that

$$\mathrm{Vol}([0,1]^n) = 1.$$

Then there is a unique $\mathbb{R}^n$-inv. measure μ on $\mathbb{R}^n/\Lambda$ such that for suitable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$:

$$\int_{\mathbb{R}^n} \varphi(x)\,dx = \int_{\mathbb{R}^n/\Lambda} d\mu(y) \sum_{\lambda \in \Lambda} \varphi(y+\lambda).$$

Let $(f_1, \dots, f_n)$ be a $\mathbb{Z}$-basis of Λ.

The set $P = \left\{ \sum_{i=1}^n x_i f_i : 0 \le x_i < 1 \right\}$ is called the fund. $\square$.

It has the property: $\forall y \in \mathbb{R}^n \ \exists ! \lambda \in \Lambda,$
$$y + \lambda \in P.$$

Thus:

~~Let~~

$$\mathrm{Vol}(\mathbb{R}^n/\Lambda) = |\det(f_1, \ldots, f_n)| = \mathrm{Vol}(P)$$

$$= \int_{\mathbb{R}^n} \chi_P(x)\,dx = \int_{\mathbb{R}^n/\Lambda} d\mu(y) \underbrace{\sum_{\lambda \in \Lambda} \chi_P(y+\lambda)}_{\overset{\shortparallel}{1}}$$

$$= \mu(\mathbb{R}^n/\Lambda).$$

Now we will prove:

<u>Theorem I. 31</u> (Minkowski). Let $G \subset \mathbb{R}^n$ be bounded convex such that $G = -G$.

Assume $\mathrm{Vol}(G) > 2^n \,\mathrm{Vol}(\mathbb{R}^n/\Lambda)$.

Then $G \cap (\Lambda \setminus (0)) \neq \emptyset$.

<u>Proof</u>: Let $B = \frac{1}{2} G$; so that B is convex, $B = -B$.

We have

$$\mathrm{Vol}(\mathbb{R}^n/\Lambda) < \mathrm{Vol}(B) = \int_{\mathbb{R}^n} \chi_B(x)\,dx$$

$$= \int_{\mathbb{R}^n/\Lambda} d\mu(y) \sum_{\lambda\in\Lambda} \chi_B(y+\lambda)$$

Thus: $\exists\, y \in \mathbb{R}^n : \sum_{\lambda\in\Lambda} \chi_B(y+\lambda) \geq 2.$

That is: $\exists\, \lambda_1 \neq \lambda_2$ in Λ, $y+\lambda_1 \in B$
$y+\lambda_2 \in B$.

Thus: $\lambda_1 - \lambda_2 = (y+\lambda_1) - (y+\lambda_2) \in B - B$

$$= \frac{1}{2}C + \frac{1}{2}C \subset C. \qquad \blacksquare$$

<u>Example I.32</u>: (the finiteness of the measure of $SL(2,\mathbb{R})/_{SL(2,\mathbb{Z})}$).

Since $SL(2,\mathbb{R})$ and $SL(2,\mathbb{Z})$ are unimodular, there is an $SL(2,\mathbb{R})$-invariant measure on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ and our objective is to show that it is finite.
This is ~~to~~ a consequence of a formula due to C.L. Siegel and that we state now.
Let $\Lambda < \mathbb{R}^2$ be a lattice in $\mathbb{R}^2$. We say $\lambda \in \Lambda$ is primitive if $\lambda \neq 0$ and if $\lambda = n\cdot\mu$, $n \in \mathbb{Z}$, $\mu \in \Lambda$ implies $n = \pm 1$. Let $\Lambda_{pr} \subset \Lambda$ be the subset of primitive vectors. For example:

$$(\mathbb{Z}^2)_{pr} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2,\ \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } (a,b) \text{ are coprime} \right\}.$$

Recall that we have an identification

$$SL(2,\mathbb{R})/_{SL(2,\mathbb{Z})} \xrightarrow{\sim} \mathcal{R}_1 \quad \text{where}$$

$$\mathcal{R}_1 = \{ \Lambda < \mathbb{R}^2 : \text{is a lattice with } \mathrm{Vol}(\mathbb{R}^2/\Lambda) = 1 \}.$$

Given $f : \mathbb{R}^2 \to \mathbb{R}$, let

$$F : \mathcal{R}_1 \longrightarrow \mathbb{R}$$

by $F(\Lambda) = \sum_{\lambda \in \Lambda_{pr}}' f(\lambda)$.

Then

<u>Theorem I.33</u> There is a choice of $SL(2,\mathbb{R})$-invariant measure on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ such that for every Borel function

$$f : \mathbb{R}^2 \longrightarrow [0,\infty)$$

we have:

$$\int_{SL(2,\mathbb{R})/SL(2,\mathbb{Z})} F(g\mathbb{Z}^2)\, d\mu(g) = \int_{\mathbb{R}^2} f(x)\, dx,$$

where f and F are related by the above formula.

Corollary I.34: $\mu\left(SL(2,\mathbb{R}) / SL(2,\mathbb{Z}) \right) < +\infty$.

Proof: Let $f = \chi_C$ where say

$C = [-1,1]^2$, so that $\mathrm{Vol}(C) = 4 > 2 \cdot 1$
$= 2 \cdot \mathrm{Vol}\left(\mathbb{R}^2 / \Lambda \right)$

for any $\Lambda \in \mathcal{R}_1$. By Minkowski's theorem,

for every $\Lambda \in \mathcal{R}_1$: $(\Lambda \setminus \{0\}) \cap C \neq \phi$.

But if $\lambda \in C \cap \Lambda$, then $\exists\, \mu \in \Lambda_{prim}$ and $m \in \mathbb{Z}$, $m \neq 0$, with $m\mu = \lambda$.

But then $\mu = \frac{1}{m}\lambda + (1 - \frac{1}{m})\, 0 \in C$

hence $\Lambda_{prim} \cap C \neq \phi$. This implies that $F(\Lambda) \geq 1$ for all $\Lambda \in \mathcal{R}_1$

and hence

$$\mu\left(SL(2,\mathbb{R}) / SL(2,\mathbb{Z}) \right) \leq \int_{SL(2,\mathbb{R})/SL(2,\mathbb{Z})} d\mu(g)\, F(g\mathbb{Z}^2) = \mathrm{Vol}(C) = 4. \quad \boxtimes$$

<u>Proof of the Thm.</u>

We will apply Cor. I.29 to two different situations.

Let $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$

$$N(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Z} \right\}.$$

Let dx be the Lebesgue measure on $\mathbb{R}^2$: it is $SL(2,\mathbb{R})$-invariant. The orbit map

$$SL(2,\mathbb{R})/_N \longrightarrow \mathbb{R}^2 \setminus (0)$$
$$\dot{g} \longmapsto \dot{g}(e_1)$$

is a homeo. and hence there is a unique $SL(2,\mathbb{R})$-invariant measure ν on $SL(2,\mathbb{R})/_N$ s.t. for any $f : \mathbb{R}^2 \longrightarrow [0,\infty)$ Borel,

$$\int_{SL(2,\mathbb{R})/_N} f(\dot{g}(e_1))\, d\nu(\dot{g}) = \int_{\mathbb{R}^2} f(x)\, dx.$$

Now let α be the unique N-invariant measure on $N/_{N(\mathbb{Z})}$ of total mesure 1.

Then there is a unique $SL(2,\mathbb{R})$-inv. measure λ on $SL(2,\mathbb{R})/_{N(\mathbb{Z})}$ such that for suitable functions:

$$\varphi : SL(2,\mathbb{R})/_{N(\mathbb{Z})} \longrightarrow [0,\infty)$$

we have:

$$\int_{SL(2,\mathbb{R})/_{N(\mathbb{Z})}} \varphi(\dot{h})\, d\lambda(\dot{h}) = \int_{SL(2,\mathbb{R})/_{N}} d\nu(\dot{g}) \int_{N/_{N(\mathbb{Z})}} \varphi(g\dot{n})\, d\alpha(\dot{n})$$

Now apply this to $\varphi(\dot{h}) = f(h(e_1))$

and observe : $\cancel{\varphi(g\dot{n}) = f(e}$

$$\varphi(g\dot{n}) = f(gn(e_1)) = f(g(e_1)).$$

Hence, since $\alpha(N/_{N(\mathbb{Z})}) = 1$:

$$\int_{SL(2,\mathbb{R})/_{N(\mathbb{Z})}} f(\dot{h}(e_1))\, d\lambda(\dot{h}) = \int_{SL(2,\mathbb{R})/_{N}} f(g(e_1))\, d\nu(\dot{g}).$$

Now we apply again Cor. I.29 to:

$$N(\mathbb{Z}) < SL(2,\mathbb{Z}) < SL(2,\mathbb{R}).$$

Then there is exactly one choice of $SL(2,\mathbb{R})$ – inv. measure on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ such that $\forall \varphi : SL(2,\mathbb{R})/N(\mathbb{Z}) \longrightarrow [0,\infty)$:

$$\int_{SL(2,\mathbb{R})/N(\mathbb{Z})} \varphi(\dot{h})\, d\lambda(\dot{h}) = \int_{SL(2,\mathbb{R})/SL(2,\mathbb{Z})} d\mu(\dot{g}) \sum_{\gamma \in SL(2,\mathbb{Z})/N(\mathbb{Z})}{}' \varphi(\dot{g\gamma})$$

Applying this to $\dot{h} \rightarrow f(h(e_1))$ we obtain:

$$\int_{SL(2,\mathbb{R})/N(\mathbb{Z})} f(h(e_1))\, d\lambda(\dot{h}) = \int_{SL(2,\mathbb{R})/SL(2,\mathbb{Z})} d\mu(\dot{g}) \sum_{\gamma \in SL(2,\mathbb{Z})/N(\mathbb{Z})}{}' f(g\gamma(e_1)).$$

Now observe that the $SL(2,\mathbb{Z})$ – orbit of $\binom{1}{0}$ coincides with $\left\{ \binom{a}{b} \in \mathbb{Z}^2 : \gcd(a,b) = 1 \right\}$

that is:

$$\sum_{\gamma \in SL(2,\mathbb{Z})/N(\mathbb{Z})}{}' f(g\gamma(e_1)) = \sum_{\lambda \in (g\mathbb{Z}^2)_{pr}}{}' f(\lambda)$$

$$= F(g\mathbb{Z}^2).$$

In fact the examples of lattices in $\mathbb{R}^n$ and $SL(2,\mathbb{Z})$ in $SL(2,\mathbb{R})$ motivates the following definition:

<u>Def. I.35</u> A lattice in a l.c.H. group G is a subgroup $\Gamma < G$ such that

(1) Γ is discrete.

(2) There is a finite G-invariant regular Borel measure on G/Γ.

$SL(n,\mathbb{Z}) < SL(n,\mathbb{R})$:

$$N = \underset{SL(n,\mathbb{R})}{\mathrm{Stab}}(e_1) = \left\{ \begin{pmatrix} 1 & v \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} : v \in \mathbb{R}^{n-1},\ A \in SL(n-1,\mathbb{R}) \right\}.$$

Then $N(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & v \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} : \begin{matrix} v \in \mathbb{Z}^{n-1} \\ A \in SL(n-1,\mathbb{Z}) \end{matrix} \right\}.$

Let α be an N-invariant measure on $N/N(\mathbb{Z})$.

Then the arguments in the proof of Thm Tm I.33 show that:

(H_n) If ~~Vol~~ $\alpha(N/N(\mathbb{Z})) < +\infty$

then there is a $SL(n,\mathbb{R})$ invariant measure μ on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ such that

$$\int_{SL(n,\mathbb{R})/SL(n,\mathbb{Z})} \hat{F}(g\mathbb{Z}^n)\, d\mu(g) = \int_{\mathbb{R}^n} f(x)\, dx$$

where $F(\Lambda) = \sum_{\lambda \in \Lambda_{prim}} f(\lambda)$.

In fact on can show that if $SL(n-1,\mathbb{Z})$ is a lattice in $SL(n-1,\mathbb{R})$, then (H_n) is satisfied.

Corollary I.36 $SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$.

We finish this chapter with an application of the regularity of left Haar measure, due to Mackey:

Thm. I.37 (Mackey) Let G, H be l.c. H. second countable groups. If $\varphi : G \to H$ is a group homomorphism that is μ_G-measurable then φ is continuous.