Then $h = \exp v$ for some $v \in B(0, r/2)$.
Now $0 < \|v\| < r/2$ and hence there $\exists$
$n \in \mathbb{N}^*$ with $r/2 < n\|v\| < r$.
But then $\underbrace{\exp(nv)}_{\exp(v)^n \in H} \in \exp(B(0,r)) \setminus U$
a contradiction. $\boxed{\xi}$

II.5. Cartan's theorem on closed subgroups.

Thm. II.54 Let G be a Lie group and $H < G$ a closed subgroup. Then H is a regular submanifold and hence a Lie group.

Lemma II.5 Let $\mathfrak{g} = V \oplus W$ be a direct sum decomposition as vector space and π_V, π_W the canonical projections on V, W. Then

$$\varphi : \mathfrak{g} \longrightarrow G$$
$$\xi \longmapsto \exp_G \pi_V(\xi) \exp_G \pi_W(\xi)$$

satisfies $D_0 \varphi = Id_{\mathfrak{g}}$.

Proof: Write φ as the composition of the following maps:

$$L : \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, \quad \xi \longmapsto (\pi_A(\xi), \pi_B(\xi))$$
$$\exp_{G \times G} : \mathfrak{g} \times \mathfrak{g} \longrightarrow G \times G, \quad (X, Y) \longmapsto (\exp_G(X), \exp_G(Y))$$

and $m : G \times G \longrightarrow G$
$$(g, h) \longmapsto gh.$$

We have $D_0 L = L$, $D_{(0,0)} \exp_{G \times G} = Id_{\mathfrak{g} \times \mathfrak{g}}$

and $D_{(0,0)} m(X, Y) = X + Y$. Thus:

$$D_0 \varphi(\xi) = \pi_A(\xi) + \pi_B(\xi) = \xi. \quad \square$$

Proof of Thm II.58:

$\exp_G^{-1}(H)$ is a closed subset of $\mathfrak{g}$ and we consider its "tangent cône" at 0:
fix a norm $\|\ \|$ on $\mathfrak{g}$ and let $\pi : \mathfrak{g}\setminus(0) \to S$ be
$\cancel{W = \{0\} \cup \{\xi \in \mathfrak{g}\setminus(0)}$
the projection on the unit sphere S.

$$W = \{0\} \cup \Big\{ \xi \in \mathfrak{g}\setminus(0) : \text{there exists a sequence } (v_n)_{n\geq 1} \text{ in } \exp_G^{-1}(H)\setminus(0) \text{ with } \lim v_n = 0 \text{ and } \lim_n \pi(v_n) = \pi(\xi) \Big\}.$$

[Picture] Observe that $\forall \xi \in W$, $\forall t \in \mathbb{R}$, $t\xi \in W$.

(1) $\exp_G(W) \subset H$: $\exp_G(0) = e \in H$.

Let $\xi \in W$, $\xi \neq 0$; ~~observe that~~ and let $(v_n)_{n\geq 1}$ be as in the definition of W:

$$\lim_{n\to\infty} \frac{v_n}{\|v_n\|} = \frac{\xi}{\|\xi\|} \quad \text{and} \quad \lim v_n = 0.$$

That is $\xi = \lim_n \frac{\|\xi\|}{\|v_n\|} v_n$

Let $a_n = \left[\frac{\|\xi\|}{\|v_n\|} \right]$ be the integer part of $\frac{\|\xi\|}{\|v_n\|}$. We claim:

$$\xi = \lim_{n \to \infty} a_n v_n .$$

Indeed: $\left\| \frac{\|\xi\|}{\|v_n\|} v_n - a_n v_n \right\| = \left\| \left(\frac{\|\xi\|}{\|v_n\|} - a_n \right) v_n \right\|$

$$\leq \|v_n\| \longrightarrow 0.$$

Thus $\exp_c(\xi) = \lim \exp_c(a_n v_n)$

$$= \lim \underbrace{(\exp_c(v_n))^{a_n}}_{\in H}$$

is in H since H is closed.

(2) W is a vector subspace of $\mathfrak{g}$:

Let $\xi \neq 0$, $\eta \neq 0$, $\xi + \eta \neq 0$ (WLOG).

Then $\exp t\xi \exp t\eta \in H$ and we consider

$u(t) := \exp_c^{-1}(\exp t\xi \exp t\eta) \in \mathfrak{g}$.

Since $D_0 \exp_G = Id$ we have:

$$u'(0) = \xi + \eta .$$

Thus: $\xi + \eta = \lim_{n \to \infty} \frac{u(1/n)}{1/n}$, and ~~$\exp_G(u(1/n))$~~ ~~$\in H$~~

~~$u(1/n) \in W$. This shows~~ $u(1/n) \in \exp_G^{-1}(H) \setminus (0)$.

This shows $\xi + \eta \in W$.

(3). There is $U \ni 0$ open subset of $\mathfrak{g}$ and a diffeo $\Phi : U \to \Phi(U) \subset G$ onto an open subset of G s.t. $\Phi(U \cap W) = \Phi(U) \cap H$.

Let W' be a complement of W:

$$\mathfrak{g} = W + W'$$

and let $V \ni 0$ be open such that

$$\begin{aligned} \Phi : V &\longrightarrow G \\ \xi &\mapsto \exp \pi_W(\xi) \exp \pi_{W'}(\xi) \end{aligned}$$

is a diffeo onto an open subset of G.

Observe that $\Phi(V \cap W) \subset H \cap \Phi(V)$.

Assume by contradiction that there is a decreasing sequence $0 \in U_n \subset U_{n-1} \subset V$ of open subsets such that

$$(1)\ \Phi(U_n \cap W) \subsetneq \Phi(U_n) \cap H$$

$$(2)\ \bigcap_{n=1}^{\infty} U_n = \{0\}.$$

Thus there is $u_n = v_n + v_n' \in U_n$ with $v_n \in W$, $v_n' \in W'$ ~~and~~ $v_n' \neq 0$ and

$$\exp_G(v_n) \exp_G(v_n') \in H.$$

Thus $\exp_G(v_n') \in H \quad \forall n \geq 1$.

Clearly $\lim v_n' = 0$ and we may assume

$$\lim_{n \to \infty} \pi(v_n') = \xi.$$

But then $\xi \in W \cap W' = (0)$ and $\xi \neq 0$ a contradiction.

(4) Thus we get a local chart at 0 and hence everywhere $\boxed{\xi}$

Here is a Corollary:

Cor. II.56 Let $\varphi : G \to H$ be a continuous homomorphism of Lie groups. Then φ is smooth.

Proof: $\operatorname{Graph} \varphi = \{(g, \varphi(g)) : g \in G\}$ is a closed subgroup of $G \times H$ hence a smooth submanifold. The projection $p_G : G \times H \to G$ induces a smooth bijective homom.

$$p_G \Big|_{\operatorname{graph}(\varphi)} \operatorname{Graph}(\varphi) \longrightarrow G . \quad (*)$$

Observe that if $\psi : L \to G$ is a smooth hom., it follows from $\psi(\exp_L tx) = \exp_G(t D_e\psi(x))$ that if ψ is injective, $D_e\psi$ is as well. This implies that $(*)$ has a smooth inverse and hence φ is smooth as the composition of:

$$G \xrightarrow{(p_G|_{\operatorname{graph}(\varphi)})^{-1}} \operatorname{Graph}(\varphi) \xrightarrow{p_H} H \qquad \boxed{\S}$$

The exponential map was an essential tool to show that closed subgroups of Lie groups are Lie groups. It is a tool as well in the following:

Thm II.57. Let G be a Lie group and $H<G$ a closed subgroup. Then G/H admits the structure of a smooth manifold such that the action $G \times G/H \to G/H$ is smooth and the projection $\pi: G \longrightarrow G/H$ is a fibration.

The idea of proof is: choose a complement $\mathfrak{f} \oplus \mathfrak{h} = \mathfrak{g}$, $\mathrm{Lie}\, H = \mathfrak{h}$ and show that $\exists\, V \ni 0$ open, $V \subset \mathfrak{f}$ s.t.

$$\pi \circ \exp|_V : V \longrightarrow G/H$$

is a homeo onto an open neighborhood of $\dot{e} \in G/H$ and

$$V \times H \longrightarrow \pi^{-1}(\pi(\exp V))$$

$$(x, h) \longmapsto \exp(x) h$$

is a diffeo.

II.6. The adjoint representation.

Let G be a Lie group; $k = \mathbb{R}, \mathbb{C}$.

Def. II.58 A representation of G is a continuous hom. $\pi : G \to GL(n,k)$.

A representation is hence smooth and gives rise to a Lie algebra hom.

$$D_e\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(n,k).$$

We have then:

Lemma II.59 (1) For every $v \in V$,

$$S_v = \{ g \in G : \pi(g)v = v \}$$ is a closed subgroup and

$$\mathrm{Lie}(S_v) = \{ X \in \mathfrak{g} : d\pi(X)v = 0 \}.$$

(2) Given a subspace $W \subset V$

$S_W = \{ g \in G : \pi(g)W \subset W \}$ is a closed subgr. with Lie algebra: $= \{ X \in \mathfrak{g} : d\pi(X)(W) \subset W \}$.

Proof: (1) $\mathrm{Lie}\, S_v = \{ X \in \mathfrak{g} : \exp tX \in S_v \ \forall t \}$

$$= \{ X \in \mathfrak{g} : \underbrace{\pi(\exp tX)\, v}_{\mathrm{Exp}(t\, d\pi(X))\, v} = v \}$$

$$= \{ v \in \mathfrak{g} : d\pi(X)\, v = 0 \}.$$

(2) Analogous. ☐

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Recall that any $g \in G$ gives a smooth auto.:

$$\mathrm{int}(g) : G \longrightarrow G$$
$$h \longmapsto g h g^{-1}.$$

Since $\mathrm{int}(g)(e) = e$, $D_e\, \mathrm{int}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie algebra automorphism, and since

$$\mathrm{int}(gh) = \mathrm{int}(g)\, \mathrm{int}(h)$$

the chain rule implies that $g \longmapsto D_e\, \mathrm{int}(g)$ defines a representation of $G \longrightarrow GL(\mathfrak{g})$.

Def. II.60 (1) The adjoint representation of the Lie group G is $\mathrm{Ad}(g) := D_e\, \mathrm{int}(g)$.

(2) The adjoint representation of the Lie algebra $\mathfrak{g}$ is $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad} = D_e \mathrm{Ad}$.

By Cor. II.48 we have the following commutative diagrams:

$$\begin{tikzcd} G \arrow[r, "\mathrm{int}(g)"] & G \\ \mathfrak{g} \arrow[u, "\exp_G"] \arrow[r, "\mathrm{Ad}(g)"] & \mathfrak{g} \arrow[u, "\exp_G"'] \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp_G"] \arrow[r, "\mathrm{ad}"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\mathrm{Exp}"'] \end{tikzcd}$$

Remark II.61: $G < GL(n,\mathbb{R})$ closed subgroup

$\mathfrak{g}$ = its Lie algebra.

Then $\mathrm{Ad}(g)X = gXg^{-1}$ $X \in \mathfrak{g}$, $g \in G$.

Indeed: $M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$

$$h \longmapsto g h g^{-1}$$

is a linear map.

<u>Prop. II.62</u>: If G is a Lie group with Lie Algebra $\mathfrak{g}$: $\mathrm{ad}(X)(Y) = [X,Y]$, $X,Y \in \mathfrak{g}$.

<u>Proof:</u>

We have

$$\mathrm{Exp}\, t\, \mathrm{ad}(X) = I + t\,\mathrm{ad}(X) + \frac{t^2}{2!}(\mathrm{ad}(X))^2 + \cdots$$
$$= \mathrm{Ad}(\exp tX)$$

~~Hence~~

So $\{\mathrm{Ad}(\exp tX)(Y)\}^L(f)$

$$= Y^L(f) + t\,(\mathrm{ad}(X)(Y))^L(f) + O(t^2)$$

Thus:

$$(\mathrm{ad}(X)(Y))^L(f)(g) = \frac{d}{dt}\Big|_{t=0} \{\mathrm{Ad}(\exp tX)(Y)\}^L(f)(g)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\big(g \exp s \cdot \mathrm{Ad}(\exp tX)(Y)\big)$$

(5)

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f\left(g \exp tX \exp sY \exp(-tX)\right)$$

$$= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f\left(g \exp tX \exp sY\right)$$

$$- \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f\left(g \exp sY \exp tX\right)$$

$$= (X^L Y^L)(f)(g) - (Y^L X^L)(f)(g)$$

$$= [X,Y]^L (f)(g). \qquad \blacksquare$$

<u>Corollary II.63</u> Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is abelian iff G^0 is abelian.

~~<u>Proof:</u> ($\Rightarrow$) $\mathfrak{g}$ abelian : then $\operatorname{ad}(X) = 0$ $\forall X$~~

~~hence $\underbrace{\operatorname{Exp} t\operatorname{ad}(X)}_{\operatorname{Ad}(\exp tX)} = \operatorname{Id}$ $\forall t, \forall X$~~

~~So $\operatorname{Ad}(\exp X)(Y) = Y$ $\forall Y$~~

~~$\Rightarrow$ $\operatorname{int}(\exp X)(\exp Y) = \exp Y$ $\forall X, \forall Y$~~

Apply Lemma I.53 (1) to the adjoint rep.:

If for $X \in \mathfrak{g}$ we set

$$Z_G(X) = \{ g : \mathrm{Ad}(g) X = X \}$$

$$Z_{\mathfrak{g}}(X) = \{ Y \in \mathfrak{g} : [Y, X] = 0 \}.$$

Then $\mathrm{Lie}\, Z_G(X) = Z_{\mathfrak{g}}(X)$.

Thus $\mathfrak{g}$ is abelian $\iff$ $\mathrm{Lie}\, Z_G(X) = \mathfrak{g} \quad \forall X \in \mathfrak{g}$

$$\iff Z_G(X)^\circ = G^\circ \quad \forall X.$$

But now: if $\mathrm{Ad}(g) X = X$, we have

$$g (\exp X) g^{-1} = \exp X.$$

Thus $Z_G(X)^\circ = G^\circ \iff G^\circ$ is abelian. ∎

Next:

<u>Def.</u> A subalgebra $\mathfrak{n} < \mathfrak{g}$ is an <u>ideal</u>, denoted $\mathfrak{n} \triangleleft \mathfrak{g}$, if $\mathrm{ad}(X)(\mathfrak{n}) \subset \mathfrak{n} \quad \forall X \in \mathfrak{g}$.

Corollaire II.64 Let N be a Lie subgr. of G with Lie algebra $\mathfrak{n}$. Then

$$\mathfrak{n} \triangleleft \mathfrak{g} \iff N^\circ \text{ is a normal subgroup of } G^\circ.$$

Proof: May assume G and N connected.

Then $S_{\mathfrak{n}} = \{ g \in G : Ad(g)(\mathfrak{n}) \subset \mathfrak{n} \}$

is a closed subgroup with Lie algebra

$$\mathfrak{s}_{\mathfrak{n}} = \{ X \in \mathfrak{g} : ad(X)(\mathfrak{n}) \subset \mathfrak{n} \}.$$

Then: $\mathfrak{s}_{\mathfrak{n}} = \mathfrak{g} \iff (S_{\mathfrak{n}})^\circ = G$

Now: $Ad(g)(X) \in \mathfrak{n}$ means that

$$\exp(t Ad(g) X) \in N \quad \forall t$$

$$\iff g(\exp tX)g^{-1} \in N \quad \forall t.$$

Thus $(S_{\mathfrak{n}})^\circ = G \iff N \triangleleft G$ since N is connected. $\boxed{\Sigma}$

In Lie algebras ideals play a role analogous to the one played in group theory by normal sub.

Namely:

(1) If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra hom. then $\operatorname{Ker} \varphi$ is an ideal in $\mathfrak{g}$.

(2) If $\mathfrak{n} \triangleleft \mathfrak{g}$ is an ideal, then on the quotient vector space $\mathfrak{g}/\mathfrak{n}$ there is a Lie algebra structure such that the can. proj.

$$p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$$

is a Lie algebra hom.

This structure is given by:

$$[X+\mathfrak{n}, Y+\mathfrak{n}] := [X,Y] + \mathfrak{n} \quad (*)$$

The fact that $\mathfrak{n}$ is an ideal makes $(*)$ well defined.

Coroll. II. 65. In the situation of Cor. II. 64 if $N \triangleleft G$ is closed, $G/_N$ has a Lie group structure with Lie algebra $\mathfrak{g}/_{\mathfrak{n}}$.

Chapter III Structure Theory.

III.1. Solvable Lie algebras and Lie groups.

Let g be a Lie algebra.

Def. III.1 A derivation of g is an endomorphism $D \in gl(g)$ such that

$$D([X,Y]) = [DX,Y] + [X,DY] \quad \forall X,Y \in g.$$

Denoting $\operatorname{Der} g$ the vector space of derivations one verifies:

Lemma III.2 (1) $\operatorname{Der} g$ is a Lie subalgebra of $gl(g)$ for the bracket $[D_1,D_2] = D_1D_2 - D_2D_1$.

(2) $\forall X \in g$:- $\operatorname{ad}(X) \in \operatorname{Der}(g)$.

The first is a direct verification and the second is a restatement of the Jacobi identity.

<u>Def. III. 3.</u> A subspace $\mathfrak{z} \subset \mathfrak{g}$ is a <u>characteristic ideal</u> if $D(\mathfrak{z}) \subset \mathfrak{z}$ $\forall D \in \mathrm{Der}\, \mathfrak{g}$.

It follows from lemma III. 2. (2) that a characteristic ideal is an ideal. A fundamental example of characteristic ideal is given by:

$$[\mathfrak{g}, \mathfrak{g}] := \text{linear span of the set } \{[X, Y] : X, Y \in \mathfrak{g}\}$$

of all brackets of elements of $\mathfrak{g}$. Since for any derivation D: $D([X,Y]) = [DX, Y] + [X, DY]$, it follows that $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal of $\mathfrak{g}$. We define then inductively:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots$$

~~Def. III. 4 $\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots$ is~~

One shows easily by recurrence that $\mathfrak{g}^{(i)}$ is a characteristic ideal in $\mathfrak{g}$.

A fundamental property of $[\mathfrak{g},\mathfrak{g}]$ is:

<u>Lemma III.4</u> (1) The quotient algebra $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ is abelian.

(2) For any Lie algebra hom. $\varphi : \mathfrak{g} \to \mathfrak{a}$ with $\mathfrak{a}$ abelian, we have $\operatorname{Ker}\varphi \supset [\mathfrak{g},\mathfrak{g}]$.

<u>Proof</u>: (1) $\big[x + [\mathfrak{g},\mathfrak{g}], Y + [\mathfrak{g},\mathfrak{g}]\big] = [x,Y] + [\mathfrak{g},\mathfrak{g}]$
$= 0 + [\mathfrak{g},\mathfrak{g}]$.

(2) Since $\varphi([x,Y]) = [\varphi(x), \varphi(Y)] = 0$ $\forall x, Y \in \mathfrak{g}$, we get $[\mathfrak{g},\mathfrak{g}] \subset \operatorname{Ker}\varphi$. $\square$

<u>Def. III.5</u> (1) $\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$ is the derived series of $\mathfrak{g}$.

(2) $\mathfrak{g}$ is <u>solvable</u> if $\mathfrak{g}^{(\ell)} = 0$ for some $\ell \in \mathbb{N}^*$.

The following is a convenient criterion for solvability :

<u>Prop. III. 6</u> A Lie algebra $\mathfrak{g}$ is solvable iff there exists a descending chain of subalgs.

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{n+1} = 0$$

such that (1) $\mathfrak{g}_{i+1} \triangleleft \mathfrak{g}_i \quad 0 \leq i \leq n$

(2) $\mathfrak{g}_i / \mathfrak{g}_{i+1}$ is abelian.

<u>Proof:</u> ($\Longrightarrow$) The derived series does the job.

($\Longleftarrow$) We have ~~$\mathfrak{g}_1 [\mathfrak{g}, \mathfrak{g}] \mathfrak{g}^{(1)}$~~

$$[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1} .$$

Starting with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_1$ one concludes by recurrence that $\mathfrak{g}^{(i)} \subset \mathfrak{g}_i$ and hence $\mathfrak{g}^{(n+1)} = 0$. ∎

We deduce that subalgebras and quotients of solvable algebras are solvable.

Cor. VI. 7 ~~Let~~ (1) $\mathfrak{h}$ subalg. of $\mathfrak{g}$. Then $\mathfrak{g}$ solv. implies $\mathfrak{h}$ solvable.

(2) Let $\mathfrak{n} \triangleleft \mathfrak{g}$. Then $\mathfrak{g}$ is solvable $\Leftrightarrow$ $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ are.

Proof: (1) By recurrence $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$.

(2) ($\Rightarrow$) clear.

($\Leftarrow$) Let $\mathfrak{g}/\mathfrak{n} \supset \mathfrak{l}_1 \supset \mathfrak{l}_2 \supset \cdots \supset \mathfrak{l}_k = (0)$

be a chain of successive ideals $\mathfrak{l}_{i+1} \triangleleft \mathfrak{l}_i$ with $\mathfrak{l}_i/\mathfrak{l}_{i+1}$ abelian. Let

$$p : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{n}$$

and $\mathfrak{g}_i := p^{-1}(\mathfrak{l}_i)$. Then $\mathfrak{g}_{i+1} \triangleleft \mathfrak{g}_i$

and $\mathfrak{g}_i/\mathfrak{g}_{i+1} \simeq \mathfrak{l}_i/\mathfrak{l}_{i+1}$ which is abelian.

Now complete the chain $\mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = \mathfrak{n}$ by a chain for $\mathfrak{n}$ as in the definition of solvability. ∎

<u>Example III.8</u> Let k be any field and

$$\mathfrak{g} := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} : * \in k \right\} \subset M_{n,n}(k).$$

Then one verifies that

$$\mathfrak{g}^{(1)} = \left\{ \begin{pmatrix} 0 & * & & \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix} \right\} \quad \text{etc...}$$

so that $\mathfrak{g}^{(n)} = 0$.

<u>Def. III.8</u> Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G is solvable if $\mathfrak{g}$ is solvable.

<u>Caution</u>: There is a notion of solvability that is defined for <u>all</u> groups, e.g. for finite groups in the context of Galois theory. Fortunately both notions agree for connected Lie groups.

<u>Prop. III . 9</u> Let G be a connected Lie group. Then G is solvable iff there is a sequence of closed subgroups

$$G \supset G_1 \supset G_2 \cdots \supset G_k = (e)$$

with (1) $G_{i+1} \triangleleft G_i$

(2) G_i / G_{i+1} is abelian.

<u>Proof</u> ($\Leftarrow$) (exercise)

($\Rightarrow$) For $\mathfrak{g} \neq 0$: Define $sol(\mathfrak{g}) := k \geq 1$ s.t.

$\mathfrak{g}^{(k-1)} \neq 0$ and $\mathfrak{g}^{(k)} = 0$.

We prove ($\Rightarrow$) by recurrence on $sol(\mathfrak{g})$.

If $sol(\mathfrak{g}) = 1$, $[\mathfrak{g}, \mathfrak{g}] = 0$, i.e. $\mathfrak{g}$ is abelian and so is G, since G is connected.

Assume $k = sol(\mathfrak{g}) \geq 2$. Let G_{k-1} be the connected Lie subgroup corresponding to $\mathfrak{g}^{(k-1)}$.

Since $\mathfrak{g}^{(k-1)}$ is abelian so is G_{k-1} and since $\mathfrak{g}^{(k-1)} \triangleleft \mathfrak{g}$ we have $G_{k-1} \triangleleft G$.

Thus $H := G_{k-1}$ is abelian, normal, closed and its Lie algebra $\mathfrak{z} \supset \mathfrak{g}^{(k-1)}$.
Then the connected Lie group G/H has Lie algebra $\mathfrak{g}/\mathfrak{z}$ and since $\mathfrak{z} \supset \mathfrak{g}^{(k-1)}$ this implies $\mathrm{sol}(\mathfrak{g}/\mathfrak{z}) \leq k-1$.
Thus there are $G/H \supset L_1 \supset \dots \supset L_j = (e)$

$j \leq k-1$, closed, $L_{i+1} \triangleleft L_i$ and L_i/L_{i+1} abelian. ~~Now let~~ Define then

$$E_1 = p^{-1}(L_1), \dots, E_j = p^{-1}(L_j) = H \supset E_{j+1}$$

$\boxed{\Xi}$

There are two fundamental thms of Lie on ~~the~~ solvability, one concerning Lie groups, the other concerning Lie algebras over fields of characteristic zero.

<u>Thm III. 10</u>

Let G be a (connected) solvable Lie group, and $\pi : G \longrightarrow GL(V)$ a representation where V is a finite dimensional $\mathbb{C}$-vector space. Then there is a basis in which $\pi(G)$ consists of upper triangular matrices.

The prove will use the fundamental concept of <u>weight</u>. Let $\pi : G \to GL(V)$ be a representation where V is a $\mathbb{C}$-v.s.

<u>Def. III. 11</u>: A weight of π is a homomorphism $\chi : G \longrightarrow \mathbb{C}^\times$ such that

$$V_\chi = \{ v \in V : \pi(g) v = \chi(g) v, \forall g \in G \} \neq 0 .$$

The main step in Thm III.10 is:

<u>Thm. III.12</u> Let G be a (conn.) solvable Lie group. Then every representation

$$\pi : G \longrightarrow GL(V) , \quad \dim V \geqslant 1$$

admits a weight.

We will need the following consequence of ?

Corollary III.13 Every connected abelian Lie group L is isomorphic to $\mathbb{R}^n \times (S^1)^m$.

Proof: We saw that $\exp_L : \mathcal{L} \longrightarrow L$ induces an isom. $\mathcal{L}/\Gamma \xrightarrow{\sim} L$ where $\Gamma = \operatorname{Ker} \exp_L$ is a discrete subgroup of $\mathcal{L}$. Now ~~there exists a basis~~

either $\Gamma = (0)$ and $\mathcal{L} \simeq \mathbb{R}^\ell$ or there is a basis $e_1, \dots, e_\ell$ of $\mathcal{L}$ and $1 \leq m \leq \ell$ with $\Gamma = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_m$. Then

$$\mathcal{L}/\Gamma \simeq (\mathbb{R}/\mathbb{Z})^m \times \mathbb{R}^{\ell - m}. \quad \boxed{}$$

Lemma III.14 Assume G connected solvable. Then there is $H \triangleleft G$, closed, connected, solvable, with $\dim H = \dim G - 1$.

Proof: Take $G_1 \lhd G$ as in proposition III.9. Then by Coroll. III.13

$$G/G_1 \cong \mathbb{R}^n \times (S^1)^m$$

Let L be the inverse image in G of $\mathbb{R}^{n-1} \times (S^1)^m$ or $\mathbb{R}^n \times (S^1)^{m-1}$. Then $L \lhd G$, is closed, solvable, with $\dim L = \dim G - 1$. Now set $H := L^\circ$. ∎

Lemma III.15 $\pi : G \to GL(V)$ a repr. with derivative $d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.

For $X \in \mathfrak{g}$, v is eigenvector of $d\pi(X)$ with e.v. $\lambda \iff v$ is eigenvector of $\exp tX$ with e.v. $e^{t\lambda}$ $\forall t \in \mathbb{R}$.

Proof: Exercise. ∎

Proof of Thm III. 12

By recurrence on $\dim G$.

If $\dim G = 1$, $\mathfrak{g} = \mathbb{R} \cdot X$. Now pick $v \neq 0$ eigenvector of $d\pi(X)$ and conclude by lemma III. 15.

Assume $\dim G \geq 2$, and pick $H \triangleleft G$ as in lemma III. 14. By recurrence, let

$$\chi : H \longrightarrow \mathbb{C}^\times$$

be a weight of $\pi|_H$. We claim that

$$V_\chi = \{ v \in V : \pi(h)v = \chi(h)v \quad \forall h \in H \}$$

is G-invariant. We have for $g \in G$, $h \in H$, $v \in V_\chi$:

$$\pi(h)\pi(g)v = \pi(g)\underbrace{\pi(g^{-1}hg)v}_{\chi(g^{-1}hg)\,v}$$

$$= \chi(g^{-1}hg)\,\pi(g)v \, .$$

Thus for h fixed, $$G \longrightarrow \mathbb{C}^\times$$ $$g \longmapsto \chi(g^{-1}hg)$$

takes values in $\mathrm{Spec}(\pi(h)) \subset \mathbb{C}^\times$, finite subset. Since G is connected we have
$$\chi(g^{-1}hg) = \chi(h), \quad \forall g, \forall h.$$

Thus V_χ is $\pi(G)$-invariant. Now write
$$\mathfrak{g} = \mathbb{R}X + \mathfrak{h}$$

and let $v_0 \in V_\chi$, $v_0 \neq 0$ be an eigenvector for $d\pi(X)$ of eigenvalue λ_0.

Observe that $\forall H \in \mathfrak{h}$,
$$d\pi(H)v_0 = d\chi(H)\cdot v_0$$

~~Thus v_0 is an eigenvector $\forall Y \in \mathfrak{g}$.~~

Setting $\lambda(aX + H) = a\lambda_0 + d\chi(H)$

we have $\lambda \in \mathfrak{g}^* = \mathrm{Lin}_{\mathbb{R}}(\mathfrak{g}, \mathbb{C})$

and $\forall Y \in \mathfrak{g}$, $\pi(\exp tY)v_0 = e^{t\lambda(Y)}v_0$

which since G is connected shows that $\pi(G)\,\mathbb{C}v_0 = \mathbb{C}v_0$, which shows that π has a weight. $\boxed{\;}$

Thm III.10 Proof :

By recurrence on $\dim V$.

If $\dim V = 1$ the assertion is clear.

Assume $\dim V \geq 2$ and let $\chi : G \to \mathbb{C}^*$ be a weight of π with weight space $V_\chi \neq 0$.

Let $e_1 \in V_\chi$, $e_1 \neq 0$. Then $\mathbb{C}e_1$ is an invariant line and on $V/\mathbb{C}e_1$ we can define
$$\pi_1(g)(v + \mathbb{C}e_1) = \pi(g)v + \mathbb{C}e_1.$$
This gives a well defined representation of G in $V/\mathbb{C}e_1$. Let then $e_2, \dots, e_n$ be vectors in V such that $e_2 + \mathbb{C}e_1, \dots, e_n + \mathbb{C}e_1$ is a basis of $V/\mathbb{C}e_1$ wrt π_1 is upper triangular.

But then T is upper triangular in the basis $e_1, \dots, e_n$. $\boxed{\xi}$