1 Exercise Sheet 1

Remark. Given a topological space, unless otherwise stated, the σ-algebra taken into account will always be the Borel σ-algebra.

Exercise 1.1. *Given two probability measures $\mu, \nu \in \mathscr{P}(\mathbb{R})$ such that μ has no atoms, prove that there exists a* nondecreasing *function $T : \mathbb{R} \to \mathbb{R}$ pushing μ into ν, i.e. $T_{\#}\mu = \nu$ ($\mu(T^{-1}(B)) = \nu(B)$ for all $B \in \mathscr{B}(\mathbb{R})$).*

Show that the assumption that μ does not have atoms is necessary.

Hint:

Start by showing the result when μ and ν are both nonatomic and their support is the whole real line $\mathbb{R}$. To do that, consider the cumulative distribution functions of the two measures.

Exercise 1.2. *Find two probability measures $\mu, \nu \in \mathscr{P}(\mathbb{R})$ such that there is more than one*[1] *Borel map $T : \mathbb{R} \to \mathbb{R}$ such that $T_{\#}\mu = \nu$ which minimizes*

$$\int_{\mathbb{R}} |x - T(x)| d\mu(x)$$

Definition. Let Z be a topological space and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on Z. We say that the sequence $(\mu_n)_{n\in\mathbb{N}}$ *weakly converge* to a measure μ on Z if

$$\int_Z f d\mu_n \xrightarrow{n\to\infty} \int_Z f d\mu$$

for all bounded continuous functions f.

Exercise 1.3 (Prokhorov's theorem)**.** *Let Z be a complete and separable metric space, then $\mathcal{F} \subset \mathscr{P}(Z)$ is sequentially relatively compact with respect to the weak topology if $\mathcal{F}$ is equi-tight, i.e. for every $\epsilon > 0$ there exists $K \subset Z$ compact such that $\mu(Z \setminus K) < \epsilon$ for all $\mu \in \mathcal{F}$.*

Hint:

1. Prove the result in the case of Z compact using Banach-Alaoglu-Bourbaki theorem.
2. Prove the result in its full generality exploiting the previous result. For doing that, take an exhaustion of Z by compact sets.

Remark. Actually Prokhorov's theorem states that the "if and only if" holds, namely $\mathcal{F}$ is sequentially relatively compact with respect to the weak topology *if and only if* $\mathcal{F}$ is equi-tight. However, for our aims, we are interested only in the aforementioned implication.

[1]Here uniqueness should be understood in the μ-almost everywhere sense.