10 Exercise Sheet 10

Exercise 10.1. *Consider the cost* $c(x,y) = \lceil |x-y| \rceil$ *(the ceiling function of the distance). Prove that, for every* $\mu, \nu \in \mathscr{P}(\Omega)$ *with* $\mu \ll \mathscr{L}^d$ *and* $\Omega \subset \mathbb{R}^d$ *compact, there exists an optimal transport plan for such a cost of the form* $\gamma = \sum_{i=1}^N \gamma_i$*, where each* γ_i *is induced by a transport map and* $N \leq \mathrm{diam}(\Omega) + 1$.

Hint:
Recall that we have proven the existence of an optimal map for the L^∞-cost.

Exercise 10.2. *Find an example of two probability measures* $\mu, \nu \in \mathscr{P}(\mathbb{R})$ *with compact support and* $\mu \ll \mathscr{L}^1$ *such that there does not exist an optimal transport plan between* μ *and* ν *with respect to the cost* $c(x,y) = \lfloor |x-y| \rfloor$ *(the floor function of the distance).*

Exercise 10.3. *Find an example of two probability measures on* $\mathbb{R}^d$ *for some* $d > 0$ *such that there is more than one optimal transport map, with respect to the quadratic cost, pushing one into the other.*