2 Exercise Sheet 2

Remark. Given a Hilbert space $(X, \langle \cdot, \cdot \rangle)$, we will say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *good-convex*[2] if it is convex and lower-semicontinuous with respect to the weak topology[3].

If you know nothing about lower-semicontinuity or weak topology, it's not a problem! Indeed **a function is good-convex if and only if it is the supremum of a family of affine functions**.

This fact is nontrivial and follows from Hahn-Banach theorem. If you still don't know functional analysis just take it for granted while solving the exercises.

Definition (Convex conjugate)**.** Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we define its *convex conjugate* $f^* : X \to \mathbb{R} \cup \{\pm\infty\}$ as

$$f^*(y) := \sup_{x \in X} \langle x, y \rangle - f(x)$$

Exercise 2.1. *Given two functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ such that $f, g \not\equiv +\infty$ and $f \le g$, show that $g^* \le f^*$. Show also that f^* and g^* are good-convex functions.*

Exercise 2.2. *Compute the convex conjugate of*

1. $f(x) = \frac{1}{2} \langle x, x \rangle$;
2. $f(x) = \langle x, x_0 \rangle$, *where $x_0 \in X$ is a fixed point;*
3. *a function f defined by $f(x_0) = 0$ and $f(x) = +\infty$ for $x \ne x_0$, where $x_0 \in X$ is a fixed point;*
4. $f(x) = \frac{1}{p}|x|^p$ *if $X = \mathbb{R}$ and $1 < p < \infty$.*

Exercise 2.3. *Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a good-convex function such that $f \not\equiv +\infty$. Prove that $(f^*)^* = f$.*

Hint:

1. Show that f^* is the smallest function such that $f(x) + f^*(y) \ge \langle x, y \rangle$ and deduce $(f^*)^* \le f$;
2. Exploiting point 2 and 3 of exercise 2.2, show that $(f^*)^* \ge f$.

Definition (Subdifferential)**.** Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ such that $f(x) \in \mathbb{R}$, we define the *subdifferential* of f at x as the set

$$\partial f(x) := \{y \in X \;:\; f(x') \ge f(x) + \langle y, x' - x \rangle \quad \forall\, x' \in X\}.$$

Exercise 2.4. *If $f : X \to \mathbb{R}$ is a convex C^1 function, prove that $\partial f(x) = \{\nabla f(x)\}$.*

Exercise 2.5. *Given a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ and a function $f : X \to \mathbb{R} \cup \{+\infty\}$, prove that*

1. *$\partial f(x)$ is convex and closed (possibly empty);*

[2]This name is not standard in literature.

[3]Actually, it is equivalent to ask the lower-semicontinuity in the strong topology. Indeed, a convex function is weak lower-semicontinuous if and only if it is strongly lower-semicontinuous.

2. $y \in \partial f(x)$ *if and only if* $f(x) + f^*(y) = \langle x, y \rangle$;

3. *If* f *is good-convex and* $f \not\equiv +\infty$, *then* $y \in \partial f(x) \iff x \in \partial f^*(y)$.

Exercise 2.6 ($\star$)**.** *Consider a strictly convex* C^1 *function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty.$$

Prove that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ *is a bijection and* $f(x) + f^*(y) = \langle x, y \rangle$ *if and only if* $\nabla f(x) = y$.