3 Exercise Sheet 3

Exercise 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the translation map $T(x) = x + x_0$. For any probability measure $\mu \in \mathscr{P}(\mathbb{R}^n)$, show that T is an optimal map from μ to $T_\#(\mu)$ with respect to the cost $c(x, y) = \frac{1}{2}|x - y|^2$.*

Hint:
Consider the potential $\phi(x) = -\langle x, x_0 \rangle$ and use weak duality.

Exercise 3.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the homothety $T(x) = \lambda x$ where $\lambda > 0$. For any compactly supported probability measure $\mu \in \mathscr{P}(\mathbb{R}^n)$, show that T is an optimal map from μ to $T_\#(\mu)$ with respect to the cost $c(x, y) = \frac{1}{2}|x - y|^2$.*

Hint:
Consider the potential $\phi(x) = \frac{1}{2}(1 - \lambda)|x|^2$ and use weak duality.

Exercise 3.3 (Rockafellar)**.** *Let $\Gamma \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a subset of $\mathbb{R}^n \times \mathbb{R}^n$. Show that the following are equivalent:*

- *There exists a good-convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that Γ is contained in the graph of $\partial\varphi$, namely if $(x, y) \in \Gamma$ then $y \in \partial\varphi(x)$.*
- *The set Γ is cyclically-monotone, that is for any choice of $(x_1, y_1), \dots, (x_k, y_k) \in \Gamma$ it holds*
$$\sum_{i=1}^{n} \langle x_i, y_i \rangle \geq \sum_{i=1}^{n} \langle x_{i+1}, y_i \rangle .$$

Hint:
Apply one of the theorem of last lecture with the correct choice of the cost c.